The Kolmogorov complexity of on-line predicting odd and even bits

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Outline

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Studying a theater play

play = 2 independent monologues x, y

- Someone studies x and y.
- Alice studies x, Bob studies y.

Script

Alice	To be or not to be, …
Bob	Once upon a time …

$\sim$ The end $\sim$

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Intuitively: total studying effort remains the same.

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Alice	I love you
Bob	I love you too
Alice	I no longer love you
Bob	I'm sad
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Alice	Blabla
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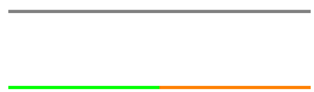
Shannon entropy: perfect symmetry of information

Splitting information “in pieces” does not increase the sum of parts of information.

$$H(X, Y) = H(X) + H(Y|X) \qquad [= H(Y) + H(X|Y)]$$



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$$H(X_1 Y_1 \cdots X_n Y_n) = \sum_n H(X_{n+1}|X_1 Y_1 \cdots X_n Y_n) + \sum_n H(Y_{n+1}|X_1 Y_1 \cdots X_{n+1})$$

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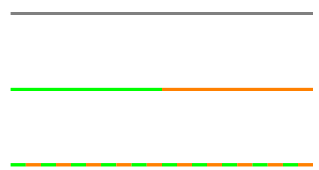

Kolmogorov complexity: almost symmetry of information

Splitting in two parts increases the sum by at most $O(\log|x|)$

$$C(x,y) = C(x) + C(y|x) + O(\log|x|)\,.$$

- Sums are not machine invariant up to $O(1)$.
- We refine to $C_{\text{odd}}(x)$ and $C_{\text{ev}}(x)$ [see further]
- Main result: $C_{\text{odd}}(x) + C_{\text{ev}}(x) \approx 2C(x)$ for infinitely many x.
  $\rightarrow$ Confirming our example.

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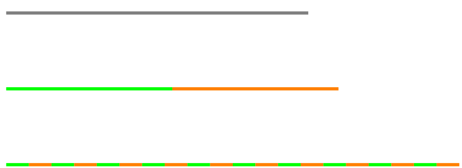
$$\begin{aligned} C(x_1y_1\cdots x_ny_n) &= \sum_n O(1) + C(x_{n+1}|x_1y_1\cdots x_ny_n) + \sum_n O(1) + C(y_{n+1}|x_1y_1\cdots x_{n+1}) \\ &= C_{\text{odd}}(x) + C_{\text{ev}}(x) + \;?? \end{aligned}$$



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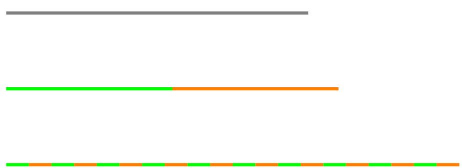

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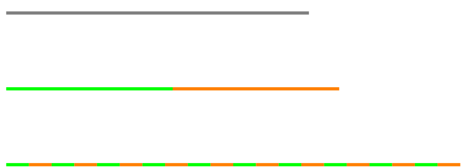

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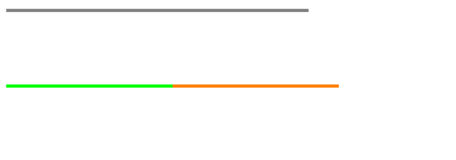

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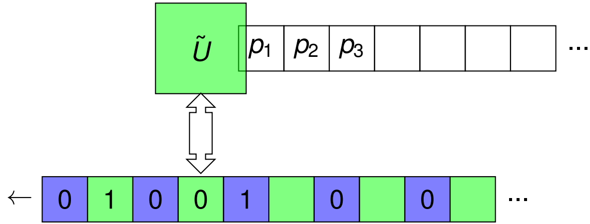


Theorem

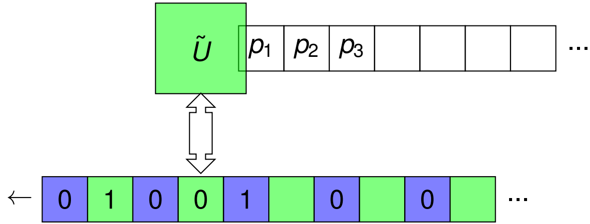
For every $\varepsilon > 0$ there exist $\delta > 0$ and a sequence ω such that for large n

$$\begin{matrix} C_{\text{odd}}(\omega_1 \ldots \omega_{2n}) \\ C_{\text{ev}}(\omega_1 \ldots \omega_{2n}) \end{matrix} \geq (1-\varepsilon)C(\omega_1 \ldots \omega_{2n}) + \delta n\,.$$

Moreover,

$$\begin{aligned} C_{\text{odd}}(\omega_2\omega_1 \ldots \omega_{2n}\omega_{2n-1}) &= C(\omega_1 \ldots \omega_{2n}) + O(\log n) \\ C_{\text{ev}}(\omega_2\omega_1 \ldots \omega_{2n}\omega_{2n-1}) &\leq O(1)\,. \end{aligned}$$

Online Kolmogorov complexity $C_{ev}(x)$



$$C(x|y) = \min\{|p| : U(p, y) = x\}$$

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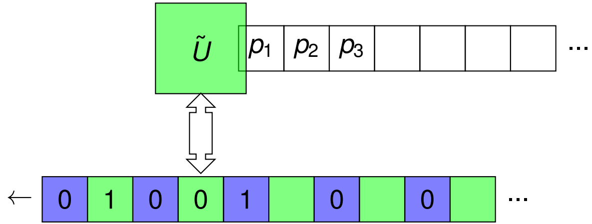

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Even Kolmogorov complexity **$C_{\text{ev}}(x)$ is**

$$\min\{|p| : U(p, x_1 \cdots x_{i-1}) = x_i, i = 2, 4, \cdots \leq |x|\}$$

Similar for C_{odd} [CSVV08].

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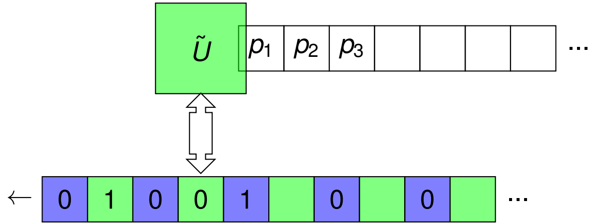

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Properties (ignoring $O(\log|x|)$-terms)

- $C_{\text{ev}}(x) \leq |x|/2$; $C_{\text{ev}}(x) \leq C(x)$
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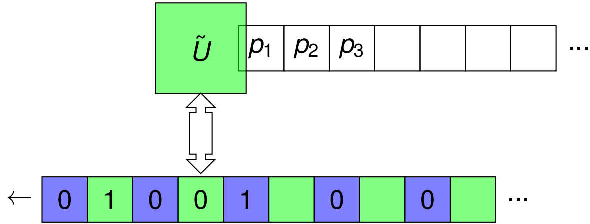

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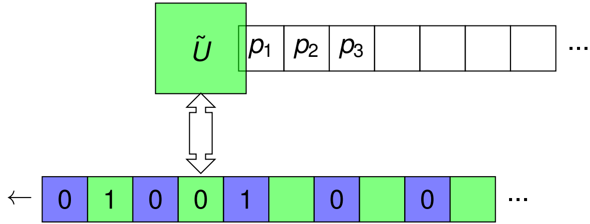

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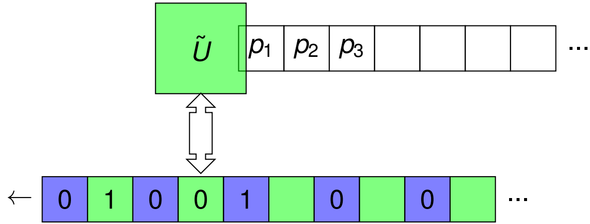

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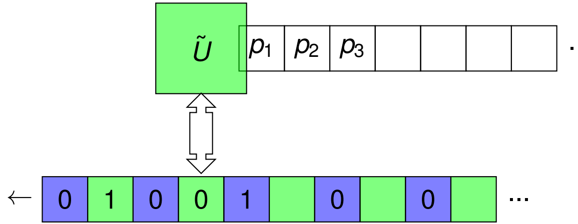

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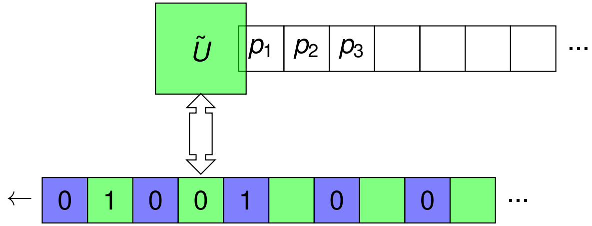

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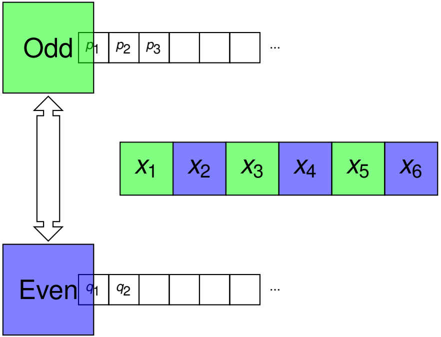

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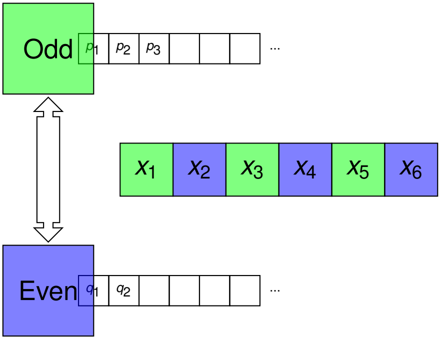

This implies:

$$\limsup_{|x|\to\infty} \frac{C_{\text{odd}}(x) + C_{\text{ev}}(x)}{C(x)} = 2\,.$$

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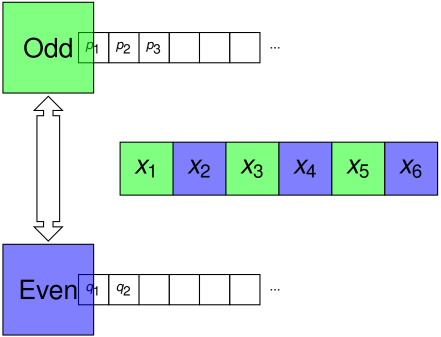


This implies:

$$\limsup_{|x|\to\infty} \frac{C_{odd}(x) + C_{ev}(x)}{C(x)} = 2 \,.$$

Measure of (instantaneous) influence
$IT(y \to x) = C(x) - C_{ev}(y_1 x_1 \cdots y_n x_n)$

$$\begin{aligned} IT(y \to x) &= C(x, y) + O(1) \\ IT(x \to y) &\leq \varepsilon C(x, y) \end{aligned}$$

Theorem

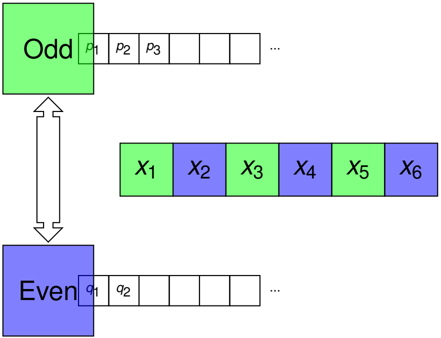
For every $\varepsilon > 0$ there exist $\delta > 0$ and a sequence ω such that for large n

$$\begin{matrix} C_{odd}(\omega_1 \ldots \omega_{2n}) \\ C_{ev}(\omega_1 \ldots \omega_{2n}) \end{matrix} \geq (1 - \varepsilon) C(\omega_1 \ldots \omega_{2n}) + \delta n \,.$$

Moreover,

$$\begin{aligned} C_{odd}(\omega_2 \omega_1 \ldots \omega_{2n} \omega_{2n-1}) &= C(\omega_1 \ldots \omega_{2n}) + O(\log n) \\ C_{ev}(\omega_2 \omega_1 \ldots \omega_{2n} \omega_{2n-1}) &\leq O(1) \,. \end{aligned}$$

Online Kolmogorov complexity $C_{\text{ev}}(x)$



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Asymmetry implies halting information

$$(C_{\text{odd}} + C_{\text{ev}})(x) - C(x) \leq (C - C^H)(x) + O(\log |x|).$$

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For every $\varepsilon > 0$ there exist $\delta > 0$ and a sequence ω such that for large n

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Generalization for more machines

$$C_{i/k} = \min\left\{|p| : U(x_1 \cdots x_{j-1}) = x_j, j = i, i+k, \ldots \leq |x|\right\}$$

x_1	x_2	x_3	x_4	x_5	x_6	···

Theorem

For each k and $\varepsilon > 0$ there exist $\delta > 0$ and a sequence ω such that for $i \leq k$ and large n

$$C_{i/k}(\omega_1 \cdots \omega_{kn}) \geq \delta n + (1-\varepsilon)C(\omega_1 \cdots \omega_{kn})$$

$$\limsup_{C(x)\to\infty} \frac{\sum_{i=1}^{k} C_{i/k}(x)}{C(x)} = k$$

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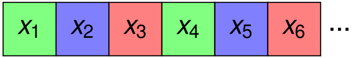

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Linear gap and upper bound

Theorem

There exist a sequence ω such that for all n

$$(C_{\text{odd}} + C_{\text{ev}} - C)(\omega_1 \ldots \omega_n) \geq n(\log \tfrac{4}{3})/2 - O(\log n)\,.$$

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There exist $\beta < \frac{1}{2}$ such that for large x

$$(C_{\text{ev}} + C_{\text{odd}} - C)\,(x) \leq \beta|x|\,.$$

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Outline

1. Introduction
2. Definitions and results
3. Application
4. Proof

Asymmetry: "Application"

- Complex system
- two regions $\mathcal{X}$ and $\mathcal{Y}$ are interacting
- measurements in each region

$$(x_1, y_1), (x_2, y_2), \ldots$$

- time resolution not enough to decide whether x_i is a reply to y_i or vice versa
- minimal description length: compare $C_{\text{odd}} + C_{\text{ev}}$ of

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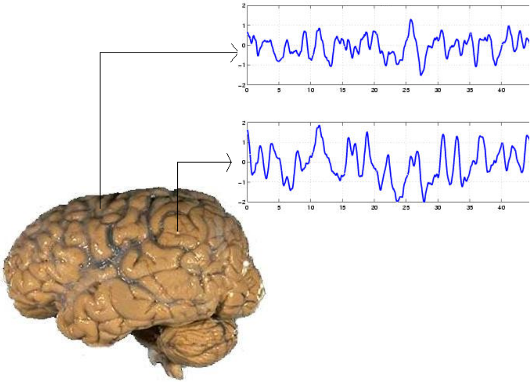

- Decompressors are non-computable, in practice always computable (or extensible to comp ones)
- But, practical ⇒ poly time computable ⇒ asymmetry is restored, even for one pair of messages
  - assume factorizing is computationally difficult
  - p, q be primes and $x_1 = (p, q)$ and $y_1 = pq$
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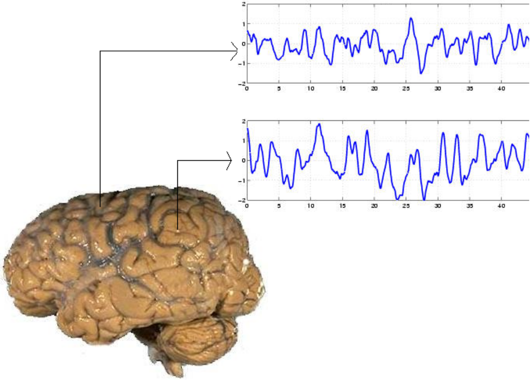

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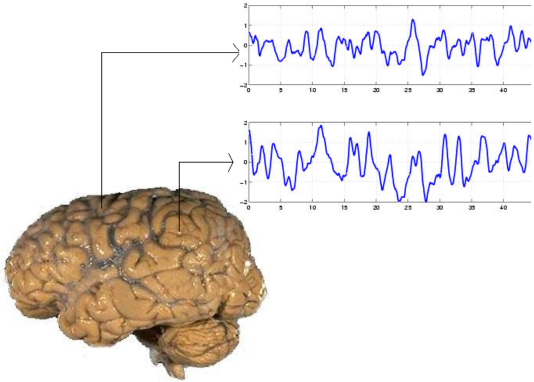

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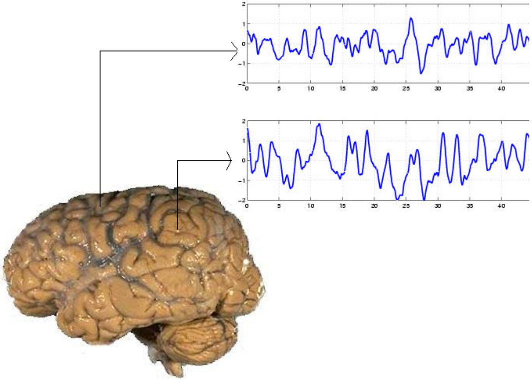

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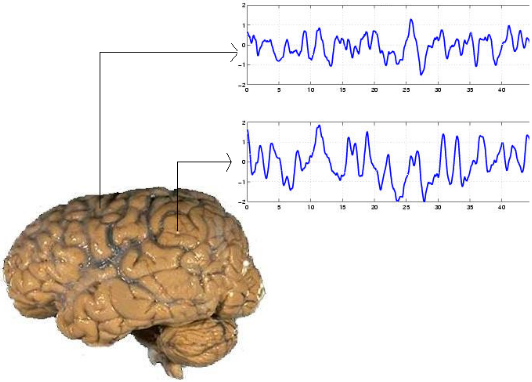

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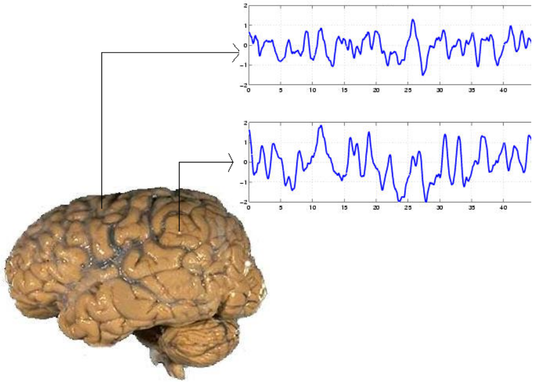

Main result: differ for some (x, y) by $O(n)$ and factor $2 - \varepsilon$.

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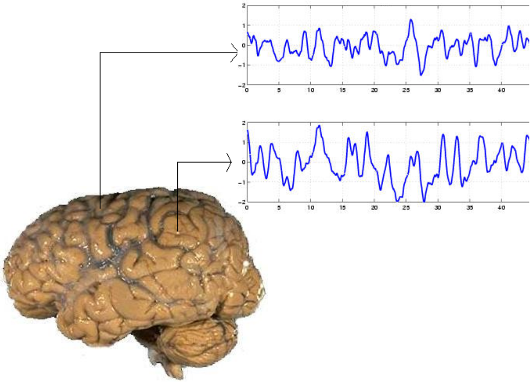

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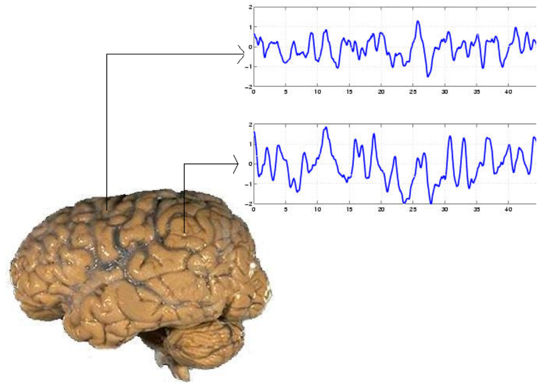

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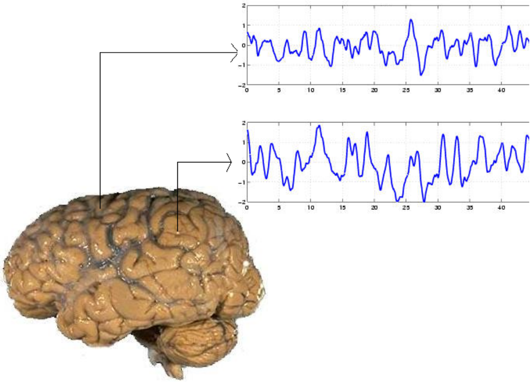

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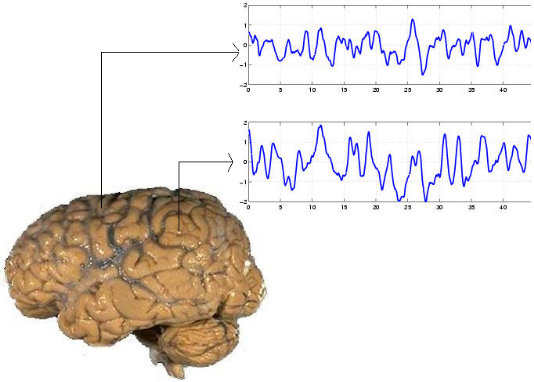

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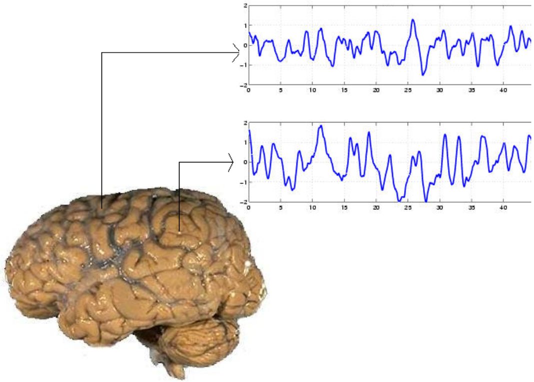

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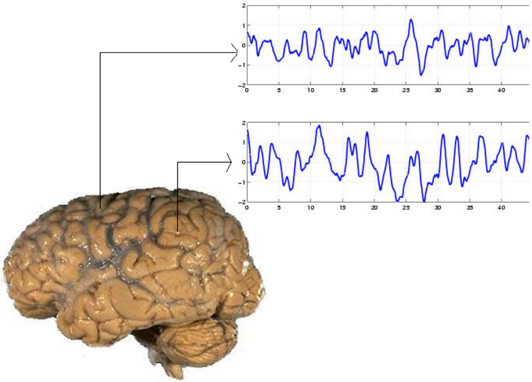

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$$H(X_n|X_{n-1}) - H(X_n|X_{n-1}, Y_{n-1})$$

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  Example where the direction means anything?

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- without: $H(X_n|X_{n-1}) - H(X_n|X_{n-1}, Y_n) \quad \to \quad$ symmetric
- algorithmic information theory: $IT(y \to x) = C(x) - C_{\text{ev}}(y_1 x_1 \cdots y_n x_n)$
- main result: for all $\varepsilon > 0$ there are x, y s.t. $IT(y \to x) = C(x, y) + O(1)$ and $IT(x \to y) \leq \varepsilon C(x, y)$,
- example with primes $\quad \to \quad$ good direction appears.

- Unfortunately, in our example no direction of influence is natural. Example where the direction means anything?

- information transfer (neurology and economics) measures influence of $\mathcal{Y}$ on $\mathcal{X}$ as

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The end, questions?

Outline

Theorem

There exist a sequence ω such that for all n

$$(C_{\text{odd}} + C_{\text{ev}})(\omega_1 \cdots \omega_{2n}) \geq n \log \tfrac{4}{3} + C(\omega_1 \cdots \omega_{2n}) + O(\log n)$$

- State the problem in terms of on-line semimeasures,
- Game on strings of length 2,
- Concatenate the winning strategies.

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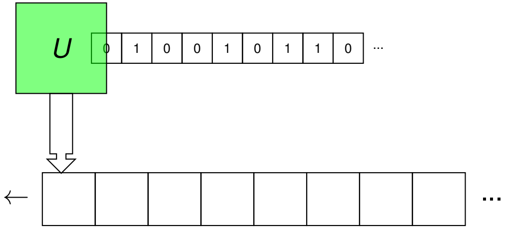
Probabilistic Turing machine $\leftrightarrow$ lower-semicomputable semimeasure



$P : \{0,1\}^* \to [0,1]$ is a semimeasure if

$$P(x0) + P(x1) \leq P(x)$$

- There exist maximal lower-semicomputable semimeasures $M(x)$.
- Coding theorem: $-\log M(x) = C(x) + O(\log C(x))$.

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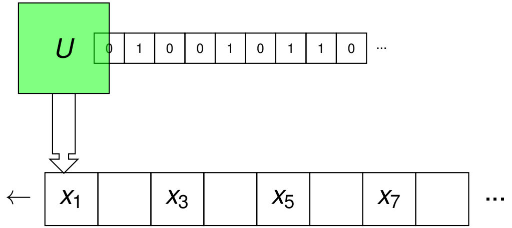

$P : \{0,1\}^* \to [0,1]$ is an even semimeasure if

$$\begin{array}{ll} P(x0) + P(x1) \le P(x) & \text{if } |x0| \text{ is even,} \\ P(x0) = P(x1) = P(x) & \text{otherwise.} \end{array}$$

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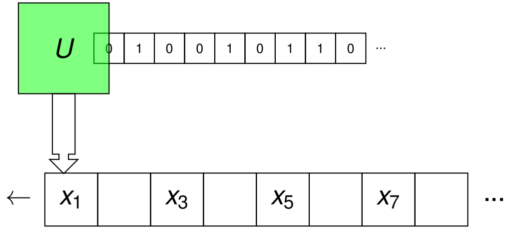

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Warning: an even machine can not be modeled by products of l.s.c. P_i.
$P_{\text{ev}}(y_1 x_1 \cdots y_n x_n) = P_1(x_1|y_1) \cdot \ldots \cdot P_n(x_n|y_n) = P(x|y)$.

For all lsc $P_{\text{odd}}, P_{\text{ev}}$ there exist ω and lsc P s.t. $(P_{\text{odd}} \cdot P_{\text{ev}})(\omega_1 \cdots \omega_{2n}) \leq \left(\frac{3}{4}\right)^n P(\omega_1 \cdots \omega_{2n})$

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Game for $|x| = 2$:

- Nature $\rightarrow P_{\text{odd}}, P_{\text{ev}}$
- Math $\rightarrow P$ s.t. $\sum_{|x|=2} P(x) \leq 3/4$

Math wins if either

- $P_{\text{odd}}(\varepsilon) > 1$
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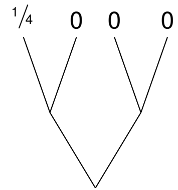

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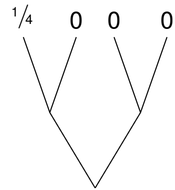

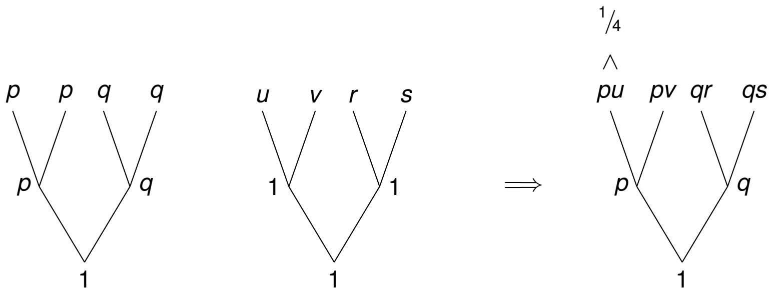

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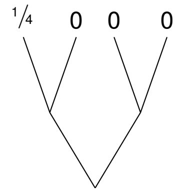

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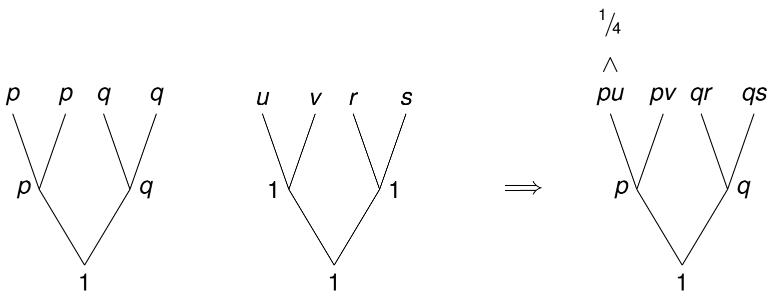
- $p > 1/2$
- $u > 1/2$

Suppose $p > 1/2$



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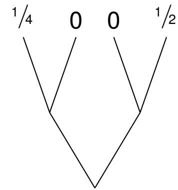

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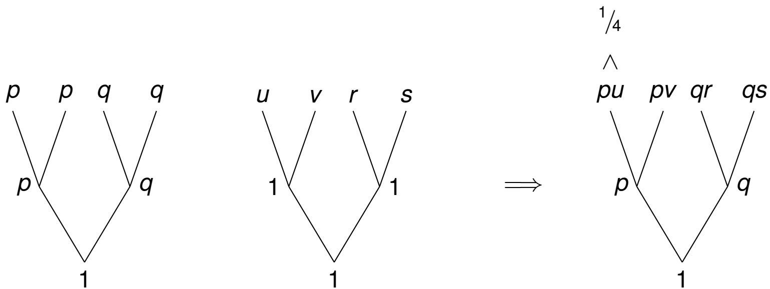

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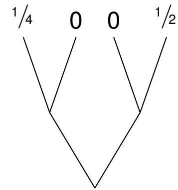

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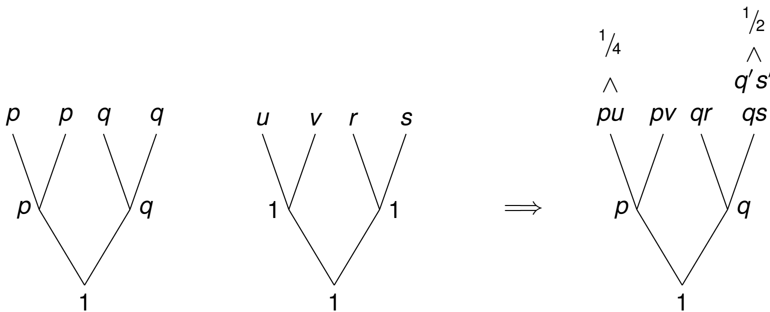
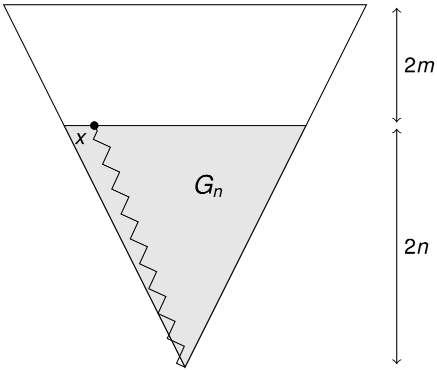


G_n is played on restrict. P_{odd}, P_{ev}
How use output in G_m?



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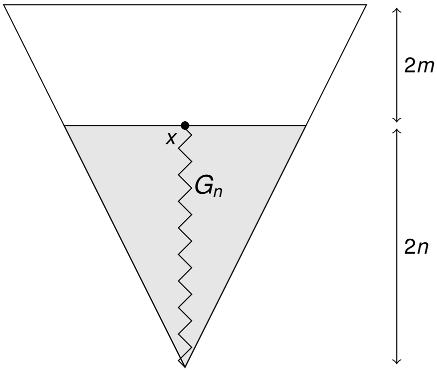

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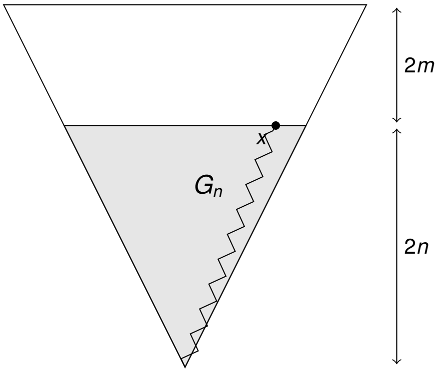

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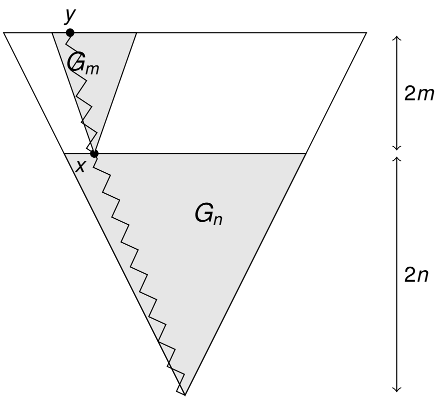

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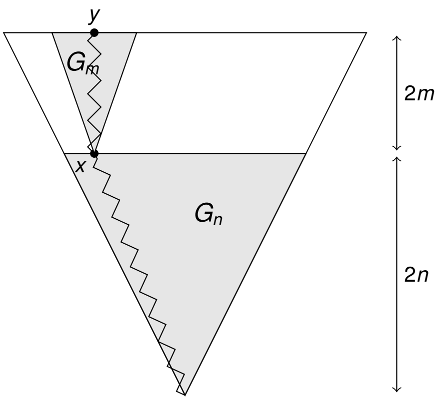

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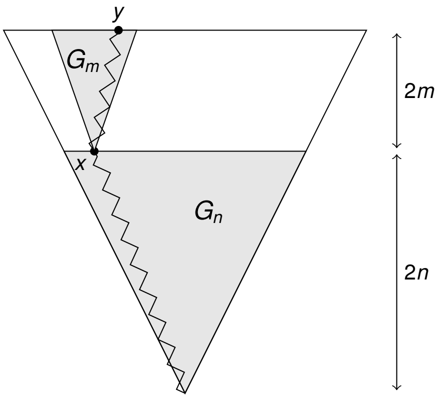

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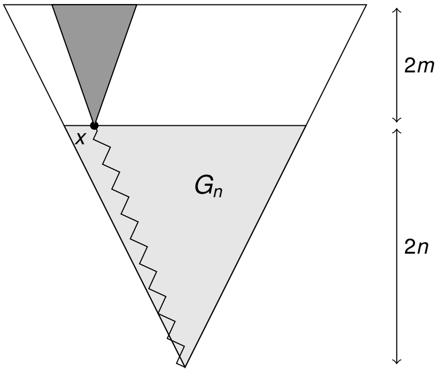

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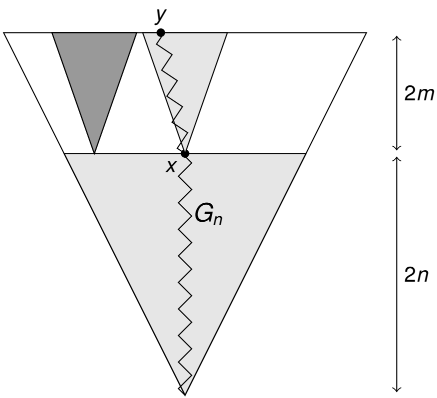

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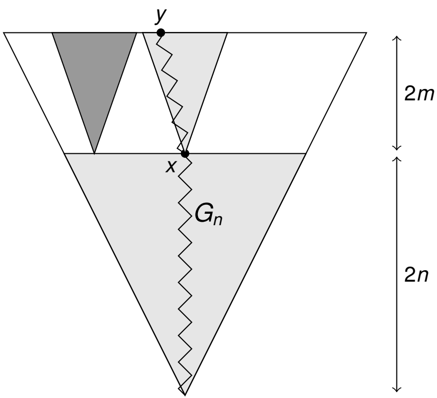

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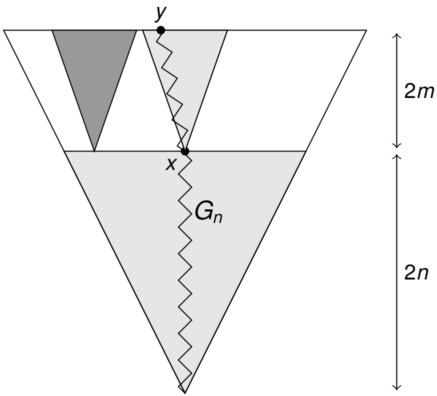

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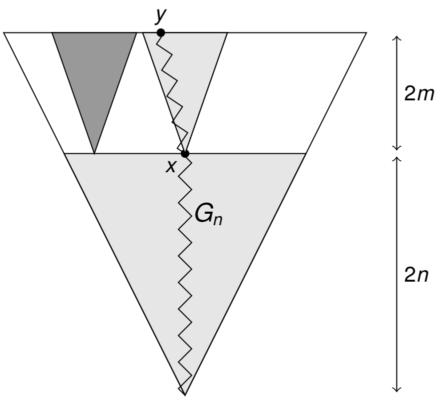

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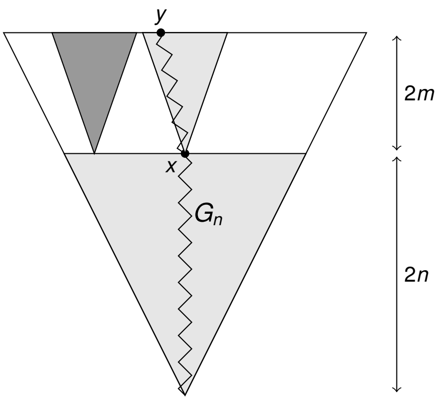

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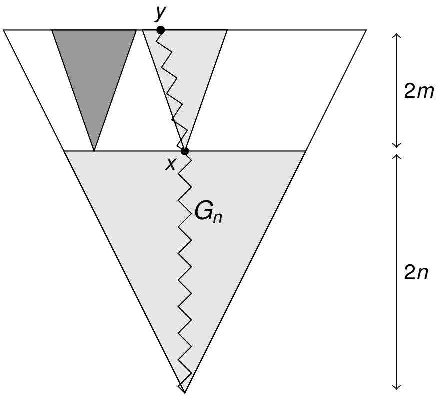

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