α-null sets: strong and weak

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Classical measure theory

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Definition (Hausdorff dimension)

Hausdorff dimension of set X: the infimum of α such that X is α-null.

Effective version

Definition (effectively null sets)

$X \subset \Omega$ is effectively null if $\exists$ algorithm M: given a rational $\varepsilon > 0$, it enumerates set of intervals $\mathcal{I} = \{I_1, \ldots, I_n, \ldots\}$:

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There exists a maximum effectively null set.

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Strong and weak α-size definitions

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Sum of sizes of all intervals in $\mathcal{I}$:

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Definition (Weak α-size)

Sum of sizes of maximal intervals in $\mathcal{I}$:

$$\mu_W^\alpha(\mathcal{I}) = \sum_{k : \nexists I' \in \mathcal{I} \supsetneq I_k} \mu^\alpha(I_k).$$

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Fact (α=1)

If $\alpha = 1$ then again these definitions are equivalent.

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Theorem (J. Reimann, F. Stephan)

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Theorem

$\forall \alpha \in \mathbb{Q}, 0 < \alpha < 1 \exists X \subset \Omega :$ *X is weak α-null but its strong α-measure is equal to 1.*

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Strong α-null sets $\subsetneq$ Solovay α-null sets $\subsetneq$ weak α-null sets.

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- We show that there exists no maximum Solovay α-null set.

Flow techniques

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- Observation

  Tree representation of Ω: cuts correspond to weak cover of X (in some sense).

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- In this game Alice can always win.

Thank you!

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- Max measure $\mu(\cup I_k)$, for all μ in some class (class of α-capacitable measures: measures such that for all intervals size determined by this measure is less then its α-size).