Normality and Finite State Dimension of Liouville Numbers

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September 2013

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Definition. [Normal Binary Sequence]

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Definition. [Normal Binary Sequence]

An infinite binary sequence ω is said to be normal in base 2 if for every natural number k and every k-long string w,

$$\lim_{n\to\infty} \frac{|\{i \in \mathbb{N} \mid \omega[i \ldots i+n-1] = w\}|}{n} = \frac{1}{2^k}.$$

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For example, the asymptotic density of 0s is 1/2,

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For example, the asymptotic density of 0s is $1/2$, the asymptotic density of 01 is $\frac{1}{4}$, and so on.

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A real number in [0,1] will be called normal if its binary expansion is a normal sequence.

Examples of Normal Numbers

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Are there normal numbers in [0,1]?

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Are there normal numbers in [0,1]?

Theorem (Borel, 1909)**.** The set of normal numbers in [0,1] has Lebesgue measure 1.

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A simple example of a normal binary sequence:

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A simple example of a normal binary sequence:

$$0\ 1\ 00\ 01\ 10\ 11$$

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This is known as the Champernowne sequence (1933).

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No normal number can be rational.

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Definition. A real number is *algebraic* if it is the root of a polynomial with integer coefficients. e.g. $\sqrt{2}$ is the root of the polynomial $x^2 - 2$.

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Are normal numbers always transcendental? (We do not know)

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Are normal numbers always transcendental? (We do not know)

Can we construct normal numbers which are provably transcendental? What can we say about normal numbers in special classes of transcendental numbers?

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Champernowne number is a transcendental number. (Mahler, 1937)

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Some transcendental numbers are far from normal.

Example. Liouville constant[1] defined as

$$\alpha = \sum_{i=0}^{\infty} \frac{1}{2^{i!}}.$$

is a transcendental number.

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is a transcendental number.

α cannot be normal - e.g. 111 never appears in α.

[1] in base 2

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Normality and Diophantine Approximation

Liouville Numbers

Theorem (Liouville, 1854)**.** A real number γ is an algebraic number of degree d. Then there is a constant C such that for every pair of natural numbers a and b,

$$\left|\gamma - \frac{a}{b}\right| < \frac{C}{b^d}.$$

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Liouville's constant is a Liouville number. (first number proved transcendental).

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Liouville numbers have Lebesgue measure 0. (e.g. Oxtoby, 1980)

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They even have Hausdorff dimension 0. (e.g. Oxtoby, 1980).
Further, they have constructive Hausdorff Dimension 0. (Staiger, 2002)

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Liouville's constant is non-normal.

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Liouville's constant is non-normal.

Surprisingly, there are normal Liouville numbers.

Idea: de Bruijn Strings

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A binary string w of length 2^n is called a de-Bruijn string of order n if every n length string occurs exactly once in w, when looked in a cyclic manner.

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$$0110$$

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Theorem (N. G. de Bruijn 1946, I. J. Good 1946)**.** Let Σ be a finite alphabet, and $k \in \mathbb{N}$. Then there exists a de Bruijn string from the alphabet Σ of order k. This string has length $|\Sigma|^k$.

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de Bruijn sequences are a standard tool in the study of normality.

Construction

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Denote the lexicographically least binary de Bruijn string of order k as $B(k)$. Consider the number defined as follows.

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β is a normal Liouville number.

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Intuition (Liouville Number): The stages have lengths approximately $k!$ as in Liouville's constant. Hence we can try to form a diophantine approximation as in Liouville's constant.

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Intuition (Liouville Number): The stages have lengths approximately $k!$ as in Liouville's constant. Hence we can try to form a diophantine approximation as in Liouville's constant.

Intution (Normality): The basic building block of each stage k is a balanced string of all orders $\leq k$.

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The length of stage k is $2^k\ k^k$.

β is a Liouville number

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The length of stage k is $2^k\ k^k$. The length of the prefix up to and including stage k is $L_k = \sum_{i=0}^{k} 2^i i^i = O((2k)^k)$.

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Consider the rational number b_k at level k defined to have the same binary expansion as β up to stage k, followed by a periodic pattern of $B(k+1)$.

β is a Liouville number

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$$|b_k - \beta| \leq \frac{1}{2^{(2(k+1))^{k+1}}} \leq \frac{1}{(2^{O((2k)^k)})^k}$$

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$$|b_k - \beta| \leq \frac{1}{2^{(2(k+1))^{k+1}}} \leq \frac{1}{(2^{O((2k)^k)})^k}$$

Hence β is a Liouville number.

β is a normal number

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In $B(k)$, every string of length k appears once (wrapping around).

β is a normal number

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For every large enough k, length$(B(k)) = o(L_{k-1})$.

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Hence for all $k \geq |w|$, the frequency of any string w is $\frac{1}{2^w}$ at the end of every block $B(k)$. What about within $B(k)$?

For every large enough k, length$(B(k)) = o(L_{k-1})$. Hence the deviations from balance within $B(k)$ is insignificant for lengths from L_{k-1} to $L_{k-1} +$ length(B_k).

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Generalization: Finite State Dimension

Consider the n-long prefix of an infinite binary sequence ω. Then for $k < n$, the frequency of a k-long word w in $\omega[0 \ldots n-1]$ is

$$\pi(\omega[0 \ldots n-1], w) = \frac{|\{i \mid 0 \leq i \leq n-k-1 \; \omega[i \ldots i+k-1] = w\}|}{n-k-1}.$$

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The entropy of this probability distribution is

$$H_{k,n}(\omega) = \sum_{w \in \Sigma^k} -\pi(\omega[0 \ldots n-1], w) \log \pi(\omega[0 \ldots n-1], w).$$

Finite-State Dimension

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The finite-dimensional entropy rate of ω is defined as $H(\omega) = \inf_k H_k(\omega)$.

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A binary sequence is normal if and only if it has finite state dimension 1 (Schnorr and Stimm 1971,

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Every binary sequence has a finite-state dimension between 0 and 1.

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Finite-state dimension thus provides a quantification of the "non-normality" of binary expansions of reals.

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Finite-state dimension thus provides a quantification of the "non-normality" of binary expansions of reals.

Can we produce Liouville numbers of any given finite-state dimension in $[0, 1]$?

Liouville numbers of any FSD between 0 and 1

Let m, n be positive integers, $m < n$. We construct a Liouville number with finite-state dimension m/n.

$$\alpha_{m/n} = 0\cdot\left((0^{2^1})^{n-m}B(1)^m\right)^{1^1}\left((0^{2^2})^{n-m}B(2)^m\right)^{2^2}\dots\left((0^{2^k})^{n-m}B(k)^m\right)^{k^k}.$$

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$\alpha_{m/n}$ is a Liouville number with finite-state dimension m/n.

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- Can we construct absolutely normal Liouville numbers? They exist. (Bugeaud, 2002)

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- Can we construct absolutely normal Liouville numbers? They exist. (Bugeaud, 2002)
- Can we construct algebraic normal numbers?

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Thank You!