Generalization of van Lambalgen's theorem and blind randomness for conditional probability

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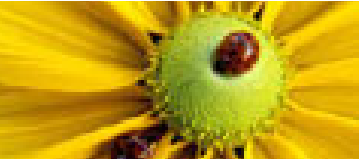


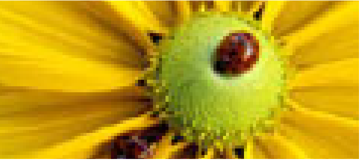
Lambalgen's theorem

Notation: $\Omega := \{0,1\}^\infty \ni x^\infty, y^\infty$, and $S := \{0,1\}^* \ni x, y$.

P_U: unifrom probability.

Theorem 1 (Lambalgen)

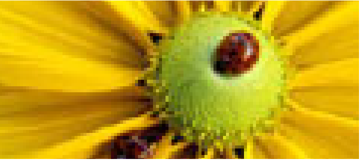
(x^∞, y^∞) *is ML-random w.r.t.* $P_U \times P_U$
$\Leftrightarrow y^\infty$ *is ML-random w.r.t.* P_U
x^∞ *is ML-random w.r.t.* P_U *relative to* y^∞.



Conditional probability

P: computable probability on $X \times Y, X = Y = \Omega$.
$P(x, y) := P(\Delta(x) \times \Delta(y)),\ \Delta(x) := \{x\omega \mid \omega \in \Omega\}$.
$P_X(x) := P(x, \Omega), P_Y(y) := (\Omega, y)$: marginal distribution

Theorem 2 (Takahashi 2008)
$P(x \mid y^\infty) := \lim_{y \to y^\infty} P(x \mid y)$ exists for all $x \in S$ and for all ML-random y^∞ w.r.t. P_Y.
$P(\cdot \mid y^\infty)$ is defined for all ML-random y^∞ w.r.t. P_Y.



Computability of conditional probability

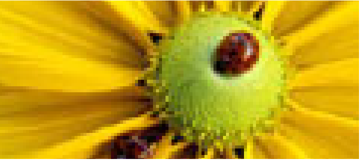
Computability of conditional probability

- uniform computability

$$\exists A \forall y^\infty \in \mathcal{R}^{P_Y} \forall x, k \exists y \sqsubset y^\infty \; |A(x,k,y) - P(x \mid y^\infty)| < \frac{1}{k}.$$

- computability for fixed y^∞.

$$\exists A \forall x, k \exists y \sqsubset y^\infty \; |A(x,k,y) - P(x \mid y^\infty)| < \frac{1}{k}.$$

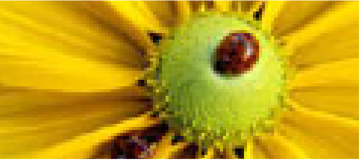


Vovk and V'yugin 93

Theorem 3 (Vovk and V'yugin) *Let P be a computable probability on $X \times Y, X = Y = \Omega$. Under the assumptions that*
(i) conditional probabilities exist for all parameters and
(ii) they are uniformly computable for all parameters,

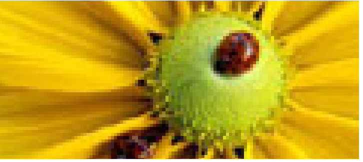
$(x^\infty, y^\infty) \in \Omega^2$ is ML-random w.r.t. P on $X \times Y$
iff y^∞ is ML-random w.r.t. P_Y and
x^∞ is ML-random w.r.t. $P(\cdot|y^\infty)$ relative to y^∞.

Theorem 4 (Roy 2011) *There is a computable probability whose conditional probability is not uniformly computable.*



Takahashi 2008, 2011

Theorem 5 (Takahashi 08, 11) *Let P be a computable probability on $X \times Y, X = Y = \Omega$. Fix a ML-random y^∞ w.r.t. P_Y. If the conditional probability $P(\cdot|y^\infty)$ is computable relative to y^∞ then,*
$(x^\infty, y^\infty) \in \Omega^2$ is ML-random w.r.t. P on $X \times Y$ iff y^∞ is ML-random w.r.t. P_Y and x^∞ is ML-random w.r.t. $P(\cdot|y^\infty)$ relative to y^∞.

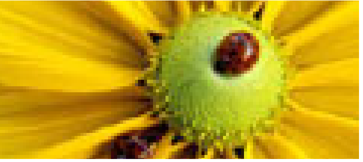


Blind randomness

P: probability on Ω.

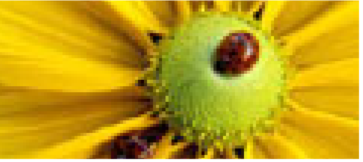
- blind test w.r.t. P: U r.e., $U_n := \{x \mid (x, n) \in U\}$, $\forall n\ U_n \supseteq U_{n+1},\ P(\tilde{U}_n) < 2^{-n}$.
- x^∞ is blind random w.r.t. P iff $x^\infty \notin \cap_n \tilde{U}_n$ for all blind test U w.r.t. P [Hanssen 10, Bienvenu et.al 11].
  If probability is not computable, the existence of universal test is not assured.
- blind test w.r.t. $P(\cdot|y^\infty)$: U r.e. set relative to y^∞, $U_n := \{x \mid (x, n) \in U\}$, $\forall n\ U_n \supseteq U_{n+1},\ P(\tilde{U}_n \mid y^\infty) < 2^{-n}$.
- x^∞ is blind random w.r.t. $P(\cdot|y^\infty)$ iff $x^\infty \notin \cap_n \tilde{U}_n$ for all blind test U w.r.t. $P(\cdot|y^\infty)$

Note: Computability assumptions on P and $P(\cdot|y^\infty)$ are not necessary.



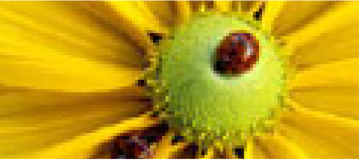
Main theorem

Theorem 6 (main theorem) *Let P be a computable probability on $X \times Y, X = Y = \Omega$. Under Assumption 1 (see below), we have $(x^\infty, y^\infty) \in \Omega^2$ is ML-random w.r.t. P on $X \times Y$ iff y^∞ is ML-random w.r.t. P_Y and x^∞ is blind-random w.r.t. $P(\cdot|y^\infty)$ relative to y^∞.*



Outline of main theorem

- The if part of the proof follows from the proof of Theorem 4.2 in [Takahashi 08] since computability of the conditional probability is not assumed in the proof.
- The proof of the only if part is similar to that of Theorem 3.3 in [Takahashi 11], however since we do not assume the computability of conditional probability, we need modify the proof.



Outline of main theorem 2

1. Fix a ML-random $\bar{y}^\infty$ w.r.t. P_Y.
2. Let $V \subseteq S$ be r.e. relative to $\bar{y}^\infty$ and $P(\tilde{V}|\bar{y}^\infty) < \epsilon$, where ϵ is a rational number.
3. From V, we construct r.e. $U \subseteq S^2$ as follows:

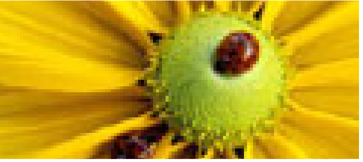
$$U \text{ is r.e., } \tilde{U}_{\bar{y}^\infty} = V, \text{ and } P(\tilde{U}) < 2\epsilon. \qquad (1)$$

4. B: partial comp.

$$V = \{x \mid \exists i, y \sqsubset y^\infty \ B(i,y) = x\}$$

$$W = \{(x,y) \mid \exists i, y \sqsubset y^\infty \ B(i,y) = x\}$$

Then W is r.e. and $\tilde{W}_{\bar{y}^\infty} = V$.

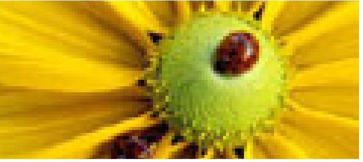


Outline of main theorem 3

For $A \subset S^2$, $\tilde{A} = \cup_{(x,y)\in A}\Delta(x,y)$.
$W' := \{(x^1,y^1),(x^2,y^2),\ldots\}$ non-overlapping r.e. s.t. $\tilde{W} = \tilde{W'}$
$W'_n = \{(x^1,y^1),\ldots(x^n,y^n)\}$
$Y_n = \{y \mid \Delta(y) \subseteq \cap_i A_i, A_i \in \{\Delta(y^i),\Delta(y^i)^c\}, 1 \le i \le n\}$

$$U_n := \{(x,y) \mid \sum_{x:(x,z)\in W'_n, z\sqsubseteq y} P(x|y) < \epsilon,\ y \in Y_n\}$$

$$U := \cup_n U_n$$



Outline of main theorem 4

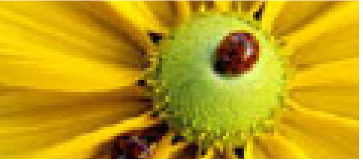
$$\widetilde{\cup_n U} = \cup_n \tilde{U}_n = \cup_n (\tilde{U}_n \setminus \tilde{U}_{n+1}) \cup \liminf_n \tilde{U}_n,$$

$$(\tilde{U}_n \setminus \tilde{U}_{n+1}) \cap (\tilde{U}_m \setminus \tilde{U}_{m+1}) = \emptyset,\ n \neq m \text{ and,}$$

$$\cup_n (\tilde{U}_n \setminus \tilde{U}_{n+1}) \cap \liminf_n \tilde{U}_n = \emptyset,$$

Then

- U is r.e.,
- $P(\liminf_n \tilde{U}_n) \leq \epsilon$,
- $V = (\liminf_n \tilde{U}_n)_{y^\infty}$.



Outline of main theorem 5

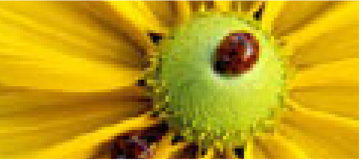
$$P(\tilde{U}) = \sum_n P(\tilde{U}_n \setminus \tilde{U}_{n+1}) + \liminf_n \tilde{U}_n < \sum_n P(\tilde{U}_n \setminus \tilde{U}_{n+1}) + \epsilon.$$

Let $f : \mathbb{Q} \to \mathbb{N}$ s.t. $\sum_{n > f(\epsilon)} P(\tilde{U}_n \setminus \tilde{U}_{n+1}) < \epsilon$.

Then $P(\tilde{U}) < 2\epsilon$.

Assumption 1: f is computable

Under the assumption $\cup_{n > f(\epsilon)} U_n$ satisfies (1).



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