On Stability Property of Probability Laws with Respect to Small Violations of Algorithmic Randomness

Vladimir V. V'yugin

Institute for Information Transmission Problems
Russian Academy of Sciences

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Martin-Löf random sequences

Martin-Löf test of randomness is a r.e. sequence $\{U_n\}$ of effectively open sets such that $P(U_n) \leq 2^{-n}$ for all n

ω passes test $\{U_n\}$ if $\omega \notin U_n$ for almost all n

ω is Martin-Löf random (w.r.to uniform L) if it passes all Martin-Löf tests

$K(x) = \min\{|p| : x \subseteq F(p))\}$ - monotonic (or prefix) complexity

$K(\omega^n) \geq n - O(1) \iff \omega$ is Martin-Löf random

We use notation: $\omega^n = \omega_1 \dots \omega_n$

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Probability laws

Pointwise form of probability law:

$$K(\omega^n) \geq n - O(1) \Longrightarrow A(\omega).$$

Law of large numbers for symmetric Bernoulli scheme:

$$K(\omega^n) \geq n - O(1) \Longrightarrow \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \omega_i = 1/2.$$

Law of iterated logarithm:

$$K(\omega^n) \geq n - O(1) \Longrightarrow \limsup_{n\to\infty} \frac{\sum_{i=1}^{n} \omega_i - n/2}{\sqrt{\frac{1}{2} n \ln \ln n}} = 1.$$

Algorithmic version of the Birkhoff's ergodic theorem

- A transformation $T : \Omega \to \Omega$ preserves a measure P if $P(T^{-1}(A)) = P(A)$ for all A.
- A measurable subset $A \subseteq \Omega$ is invariant with respect to T if $T^{-1}(A) = A$ modulo a set of measure 0.
- T is ergodic if $P(A) = 0$ or $P(A) = 1$ for each invariant A.

Theorem

For any computable transformation T preserving the uniform measure and computable bounded observable f

$$K(\omega^n) \geq n - O(1) \Rightarrow \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i\omega) = \hat{f}(\omega)$$

for some $\hat{f}$ $(= E(f)$ for ergodic $T)$.

Local stability of probability laws

Law of large numbers (Schnorr (1973):

$$K(\omega^n) \geq n - \alpha(n) - O(1) \Longrightarrow \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \omega_i = 1/2,$$

where $\alpha(n) = o(n)$ as $n \to \infty$.

Law of iterated logarithm (Vovk (1986)):

$$K(\omega^n) \geq n - \alpha(n) - O(1) \Longrightarrow \limsup_{n\to\infty} \frac{\sum_{i=1}^{n} \omega_i - n/2}{\sqrt{\frac{1}{2} n \ln\ln n}} = 1,$$

where $\alpha(n) = o(\ln\ln n)$ as $n \to \infty$.

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Sufficient condition for stability of a probability law

Why such stability?

Schnorr test of randomness is a Martin-Löf test of randomness U_n such that the measure $L(U_n)$ is the computable function of n.

Theorem

For any Schnorr test of randomness $\mathcal{T}$ a computable unbounded function $\rho(n)$ exists such that for any infinite sequence ω if $K(\omega^n) \geq n - \rho(n) - O(1)$ then the sequence ω passes the test $\mathcal{T}$.

Effective convergence almost surely (a.s.)

$f_n(\omega)$ – computable sequence of functions of type: $\Omega \to [a,b]$. $f_n(\omega) \to f(\omega)$ a.s. effectively converges if a computable function $N(\delta,\varepsilon)$ exists such that $L\{\omega : \sup_{n \geq N(\delta,\varepsilon)} |f_n(\omega) - f(\omega)| > \delta\} < \varepsilon$ for all positive rational numbers δ and ε.

Theorem

If $f_n(\omega) \to f(\omega)$ a.s. effectively converges then a Schnorr test of randomness exists such that if a sequence ω passes this test then $\lim_{n\to\infty} f_n(\omega) = f(\omega)$.

We refer this to Hoyrup, Rojas (see also Franklin, Towsner "Randomness and non-ergodic systems" http://www.math.uconn.edu/ franklin/papers/ft-ergodic.pdf)

Sufficient condition for local stability

Corollary

If $f_n(\omega)$ a.s. effectively converges to $f(\omega)$ then a computable unbounded function $\rho(n)$ exists such that for any infinite sequence ω

$$K(\omega^n) \geq n - \rho(n) - O(1) \Longrightarrow \lim_{n\to\infty} f_n(\omega) = f(\omega).$$

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Scheme of proving local stability

$f_n(\omega) \to f(\omega)$ a.s. effectively converges $\Longrightarrow$

$f_n(\omega) \to f(\omega)$ for any Schnorr random $\omega \Longrightarrow$

$f_n(\omega) \to f(\omega)$ pointwise locally stable converges

Stable laws

- SLLN: $A_n(\omega) = \frac{1}{n}\sum_{i=1}^{n} \omega_i$ a.s. effectively converges to $\frac{1}{2}$.
- For any computable ergodic transformation preserving measure L and any computable bounded observable f, $E_n(\omega) = \frac{1}{n}\sum_{k=0}^{n-1} f(T^k\omega)$ a.s. effectively converges to $\int f dL$.

1) Item 1 follows from Chernoff inequality.
2) Item 2 follows from a generalization of maximal ergodic theorem by Galatolo, Hoyrup, Rojas "Computing the speed of convergence of ergodic averages and pseudorandom points in computable dynamical systems", EPTCS 24, 2010, pp. 718,

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Local stability property for ergodic case

Theorem

For any computable ergodic transformation T preserving the uniform measure and a computable bounded function f, a computable unbounded function $\alpha(n)$ exists such that

$$K(\omega^n) \geq n - \alpha(n) - O(1) \text{ for all } n \implies \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^i\omega) = \int f dL.$$

$\alpha(n)$ depends on T and f.

Uniform instability of the ergodic theorem

We cannot define such an $\alpha(n)$ common to all ergodic T and f.

Theorem

For any nondecreasing unbounded computable function $\alpha(n)$, a computable ergodic transformation T, a computable indicator function f, and a sequence $\omega \in \Omega$ exist such that

$$K(\omega^n) \geq n - \alpha(n) \text{ for all } n \text{ and}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i \omega) \text{ does not exist.}$$

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Instability property for non-ergodic transformations

Local stability property fails for non-ergodic transformations.

Theorem

A computable transformation T preserving the uniform measure exists such that for each unbounded computable function $\alpha(n)$ an infinite sequence $\omega \in \Omega$ exists such that

$$K(\omega^n) \geq n - \alpha(n) \text{ for all } n \text{ and}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i\omega) \text{ does not exist,}$$

for some computable indicator function f.

Transformation T is non-ergodic.

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Shannon – McMillan – Breiman theorem

$T(\omega_1\omega_2\dots) = \omega_2\omega_3\dots$ – left shift;

P – ergodic if it is invariant with respect to shift T.

An algorithmic version of the Shannon – McMillan – Breiman theorem holds Hochman (2009)):

Theorem

For any computable stationary ergodic measure P

$$K(\omega^n) \geq -\log P(\omega^n) - O(1) \Longrightarrow$$
$$\lim_{n\to\infty} \frac{K(\omega^n)}{n} = \lim_{n\to\infty} \frac{-\log P(\omega^n)}{n} = H,$$

where H is the entropy of the measure P.

Uniform instability property of the SMB theorem

Theorem

For any nondecreasing unbounded computable function $\alpha(n)$ and for any and $0 < \varepsilon < 1/4$ a computable stationary ergodic measure P with entropy $0 < H \leq \varepsilon$ and an infinite binary sequence ω exist such that

$$K(\omega^n) \geq -\log P(\omega^n) - \alpha(n) \text{ for all } n,$$

$$\limsup_{n\to\infty} \frac{K(\omega^n)}{n} \geq \frac{1}{4}$$

and

$$\liminf_{n\to\infty} \frac{K(\omega^n)}{n} \leq \varepsilon.$$

Does a local stability holds for SMB theorem is an open problem.

Universal compression scheme

A code is a sequence of functions $\phi_n : \{0,1\}^n \to \{0,1\}^*$.
A code $\{\phi_n\}$ is called *universal* with respect to a class of stationary ergodic sources if for any computable stationary ergodic measure P (with entropy H_P)

$$\lim_{n\to\infty} \rho_{\phi_n}(\omega^n) = \frac{l(\phi_n(\omega^n))}{n} = H_P$$

almost surely, where $l(x)$ is length of a word x.

Lempel – Ziv coding scheme is an example of such universal coding scheme.

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Uniform instability property of any universal coding scheme

Theorem

For any unbounded nondecreasing computable function $\alpha(n)$ *and* $0 < \varepsilon < 1/4$ *a computable stationary ergodic measure* P *with entropy* $0 < H \le \varepsilon$ *exists such that for each universal code* $\{\phi_n\}$ *an infinite binary sequence* ω *exists such that*

$$K(\omega^n) \ge -\log P(\omega^n) - \alpha(n) \text{ for all } n,$$

$$\limsup_{n\to\infty} \rho_{\phi_n}(\omega^n) \ge \frac{1}{4}$$

$$\liminf_{n\to\infty} \rho_{\phi_n}(\omega^n) \le \varepsilon.$$

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