The Kolmogorov complexity of on-line predicting odd and even bits

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1 Introduction

2 Definitions and results

3 Application

4 Proof
Studying a theater play

play = 2 independent monologues \( x, y \)

- Someone studies \( x \) and \( y \).
- Alice studies \( x \), Bob studies \( y \).

\[ \text{Script} \]

Alice

To be or not to be, …

…

Bob

Once upon a time

…

\sim The end \sim
play = 2 independent monologues $x, y$

- Someone studies $x$ and $y$.
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To be or not to be,  
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~ The end ~
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- Alice studies $x$, Bob studies $y$.

\[ \text{Script} \]

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To be or not to be,  
\[ \ldots \]

Bob

Once upon a time  
\[ \ldots \]

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Studying a theater play

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- Someone studies $x$ and $y$.
- Alice studies $x$, Bob studies $y$.

Intuitively: total studying effort remains the same.

---

Script

Alice  To be or not to be,  
   ...  

Bob    Once upon a time  
   ...  

~ The end ~
Studying a theater play

Play = large dialogue, alternating lines
- Someone studies everything,
- Alice studies odd and Bob even lines.

**Script**

Alice: I love you
Bob: I love you too
Alice: I no longer love you
Bob: I’m sad

... 

Alice: Blabra
Bob: Blabra

~ The end ~
Studying a theater play

Play = large dialogue, alternating lines
- Someone studies everything,
- Alice studies odd and Bob even lines.

<table>
<thead>
<tr>
<th>Alice</th>
<th>I love you</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bob</td>
<td>I love you too</td>
</tr>
<tr>
<td>Alice</td>
<td>I no longer love you</td>
</tr>
<tr>
<td>Bob</td>
<td>I’m sad</td>
</tr>
</tbody>
</table>

... 

Alice  Blabla
Bob    Blabla

~ The end ~
Studying a theater play

Play = large dialogue, alternating lines
  - Someone studies everything,
  - Alice studies odd and Bob even lines.
Only remember lines with help of last other lines.

Script

| Alice   | I love you                  |
| Bob     | I love you too              |
| Alice   | I no longer love you        |
| Bob     | I’m sad                     |

...  

Alice  | Blabla                     |
Bob    | Blabla                     |

~ The end ~
Studying a theater play

Play = large dialogue, alternating lines
- Someone studies everything,
- Alice studies odd and Bob even lines.

Only remember lines with help of last other lines.
Intuitively: total studying effort is the same?

Script

Alice    I love you
Bob      I love you too
Alice    I no longer love you
Bob      I’m sad

... 

Alice    Blabla
Bob      Blabla

~ The end ~
Shannon entropy: perfect symmetry of information

Splitting information “in pieces” does not increase the sum of parts of information.

\[ H(X, Y) = H(X) + H(Y|X) \quad [\equiv H(Y) + H(X|Y)] \]
Shannon entropy: perfect symmetry of information

Splitting information “in pieces” does not increase the sum of parts of information.

\[ H(X, Y) = H(X) + H(Y|X) \quad [= H(Y) + H(X|Y)] \]

By recursion, for any number of “pieces”

\[ H(X_1 Y_1 \cdots X_n Y_n) = \sum_n H(X_{n+1}|X_1 Y_1 \cdots X_n Y_n) + \sum_n H(Y_{n+1}|X_1 Y_1 \cdots X_{n+1}) \]
Shannon entropy: perfect symmetry of information

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Splitting in two parts increases the sum by at most $O(\log |x|)$

$$C(x, y) = C(x) + C(y|x) + O(\log |x|).$$

- Sums are not machine invariant up to $O(1)$.
- We refine to $C_{\text{odd}}(x)$ and $C_{\text{ev}}(x)$ [see further]
- Main result: $C_{\text{odd}}(x) + C_{\text{ev}}(x) \approx 2C(x)$ for infinitely many $x$.
  $\rightarrow$ Confirming our example.
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By recursion, for $n$ “pieces” an excess $O(n)$

$$C(x_1y_1 \cdots x_ny_n) = \sum_n O(1) + C(x_{n+1}|x_1y_1 \cdots x_ny_n) + \sum_n O(1) + C(y_{n+1}|x_1y_1 \cdots x_{n+1})$$

$$= C_{\text{odd}}(x) + C_{\text{ev}}(x) + ??$$

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  $\rightarrow$ Confirming our example.
Kolmogorov complexity: almost symmetry of information

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Splitting in two parts increases the sum by at most $O(\log |x|)$

\[ C(x, y) = C(x) + C(y|x) + O(\log |x|). \]

By recursion, for $n$ “pieces” an excess $O(n)$

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C(x_1 y_1 \cdots x_n y_n) = \sum_n O(1) + C(x_{n+1}|x_1 y_1 \cdots x_n y_n) + \sum_n O(1) + C(y_{n+1}|x_1 y_1 \cdots x_{n+1})
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\[ = C_{\text{odd}}(x) + C_{\text{ev}}(x) + ?? \]

Sums are not machine invariant up to $O(1)$.

We refine to $C_{\text{odd}}(x)$ and $C_{\text{ev}}(x)$ [see further]

Main result: $C_{\text{odd}}(x) + C_{\text{ev}}(x) \approx 2C(x)$ for infinitely many $x$.

→ Confirming our example.
Outline

1 Introduction

2 Definitions and results

3 Application

4 Proof
Online Kolmogorov complexity $C_{ev}(x)$

**Theorem**

For every $\epsilon > 0$ there exist $\delta > 0$ and a sequence $\omega$ such that for large $n$

$$\frac{C_{odd}(\omega_1 \ldots \omega_{2n})}{C_{ev}(\omega_1 \ldots \omega_{2n})} \geq (1 - \epsilon)C(\omega_1 \ldots \omega_{2n}) + \delta n.$$  

Moreover,

$$C_{odd}(\omega_2 \omega_1 \ldots \omega_{2n+2}) = C(\omega_1 \ldots \omega_{2n}) + O(\log n)$$  

$$C_{ev}(\omega_2 \omega_1 \ldots \omega_{2n+2}) \leq O(1).$$
Online Kolmogorov complexity $C_{ev}(x)$

$C(x|y) = \min\{|p| : U(p, y) = x\}$

**Theorem**

For every $\varepsilon > 0$ there exist $\delta > 0$ and a sequence $\omega$ such that for large $n$

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Moreover,

$$C_{odd}(\omega_2 \omega_1 \ldots \omega_{2n} \omega_{2n-1}) = C(\omega_1 \ldots \omega_{2n}) + O(\log n)$$

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Online Kolmogorov complexity $C_{ev}(x)$

\[ C(x|y) = \min \{|p| : U(p, y) = x\} \]

**Even Kolmogorov complexity** $C_{ev}(x)$ is

\[ \min \{|p| : U(p, x_1 \cdots x_{i-1}) = x_i, i = 2, 4, \cdots \leq |x|\} \]

Similar for $C_{odd}$ [CSVV08].

**Theorem**

For every $\varepsilon > 0$ there exist $\delta > 0$ and a sequence $\omega$ such that for large $n$

\[ C_{odd}(\omega_1 \ldots \omega_{2n}) \geq (1 - \varepsilon)C(\omega_1 \ldots \omega_{2n}) + \delta n. \]

Moreover,

\[ C_{odd}(\omega_2\omega_1 \ldots \omega_{2n}\omega_{2n-1}) = C(\omega_1 \ldots \omega_{2n}) + O(\log n) \]
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Online Kolmogorov complexity $C_{ev}(x)$

**Properties (ignoring $O(\log |x|)$-terms)**

- $C_{ev}(x) \leq |x|/2$; $C_{ev}(x) \leq C(x)$
- $(C_{odd} + C_{ev})(x) - C(x) \leq |x|/2$
- $C(x) \leq (C_{odd} + C_{ev})(x) \leq 2C(x)$

**Theorem**

*For every $\varepsilon > 0$ there exist $\delta > 0$ and a sequence $\omega$ such that for large $n$*

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\frac{C_{odd}(\omega_1 \ldots \omega_{2n})}{C_{ev}(\omega_1 \ldots \omega_{2n})} \geq (1 - \varepsilon)C(\omega_1 \ldots \omega_{2n}) + \delta n.
$$

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\[
\tilde{U} \begin{array}{ccc}
p_1 & p_2 & p_3 \\
\end{array} \ldots
\]

\[
\begin{array}{c}
0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ \\
\end{array} \ldots
\]

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Similarly for $C_{odd}$ [CSVV08].

**Properties (ignoring $O(\log |x|)$-terms)**

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This implies:

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\limsup_{|x| \to \infty} \frac{C_{\text{odd}}(x) + C_{\text{ev}}(x)}{C(x)} = 2.
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**Theorem**

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*Moreover,*

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C_{\text{odd}}(\omega_2\omega_1 \ldots \omega_{2n}\omega_{2n-1}) = C(\omega_1 \ldots \omega_{2n}) + O(\log n)
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Measure of (instantaneous) influence

\[ IT(y \to x) = C(x) - C_{\text{ev}}(y_1x_1 \cdots y_nx_n) \]

\[ IT(y \to x) = C(x, y) + O(1) \]

\[ IT(x \to y) \leq \varepsilon C(x, y) \]

**Theorem**

For every \( \varepsilon > 0 \) there exist \( \delta > 0 \) and a sequence \( \omega \) such that for large \( n \)

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Measure of (instantaneous) influence

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$$IT(y \to x) = C(x, y) + O(1)$$

$$IT(x \to y) \leq \varepsilon C(x, y)$$

Asymmetry implies halting information

$$(C_{\text{odd}} + C_{\text{ev}})(x) - C(x) \leq (C - C^H)(x) + O(\log |x|).$$

**Theorem**

For every $\varepsilon > 0$ there exist $\delta > 0$ and a sequence $\omega$ such that for large $n$

$$\frac{C_{\text{odd}}(\omega_1 \ldots \omega_{2n})}{C_{\text{ev}}(\omega_1 \ldots \omega_{2n})} \geq (1 - \varepsilon)C(\omega_1 \ldots \omega_{2n}) + \delta n.$$ 

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Generalization for more machines

\[ C_{i/k} = \min \{ |p| : U(x_1 \cdots x_{j-1}) = x_j, j = i, i+k, \ldots \leq |x| \} \]

\[
\begin{array}{cccccc}
  x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \ldots \\
\end{array}
\]

**Theorem**

For each \( k \) and \( \varepsilon > 0 \) there exist \( \delta > 0 \) and a sequence \( \omega \) such that for \( i \leq k \) and large \( n \)

\[ C_{i/k}(\omega_1 \cdots \omega_{kn}) \geq \delta n + (1 - \varepsilon)C(\omega_1 \cdots \omega_{kn}) \]

\[
\limsup_{C(x) \to \infty} \frac{\sum_{i=1}^{k} C_{i/k}(x)}{C(x)} = k
\]
Generalization for more machines

\[ C_{i/k} = \min \{|p| : U(x_1 \cdots x_{j-1}) = x_j, j = i, i + k, \ldots \leq |x|\} \]

\[
\begin{array}{cccccc}
\text{x}_1 & \text{x}_2 & \text{x}_3 & \text{x}_4 & \text{x}_5 & \text{x}_6 & \ldots
\end{array}
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Theorem

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Generalization for more machines

\[ C_{i/k} = \min \{|p| : U(x_1 \cdots x_{j-1}) = x_j, j = i, i+k, \ldots \leq |x| \} \]

\[ x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad \ldots \]

**Theorem**

*For each \( k \) and \( \varepsilon > 0 \) there exist \( \delta > 0 \) and a sequence \( \omega \) such that for \( i \leq k \) and large \( n \)

\[ C_{i/k}(\omega_1 \cdots \omega_{kn}) \geq \delta n + (1 - \varepsilon)C(\omega_1 \cdots \omega_{kn}) \]

\[ \limsup_{C(x) \to \infty} \frac{\sum_{i=1}^{k} C_{i/k}(x)}{C(x)} = k \]
There exist a sequence $\omega$ such that for all $n$

$$(C_{\text{odd}} + C_{\text{ev}} - C)(\omega_1 \ldots \omega_n) \geq n(\log \frac{4}{3})/2 - O(\log n).$$

Moreover,

$$C_{\text{odd}}(\omega_2\omega_1 \ldots \omega_{2n}\omega_{2n-1}) = C(\omega_1 \ldots \omega_{2n}) + O(\log n)$$

and

$$C_{\text{ev}}(\omega_2\omega_1 \ldots \omega_{2n}\omega_{2n-1}) \leq O(1).$$

There exist $\beta < \frac{1}{2}$ such that for large $x$

$$(C_{\text{ev}} + C_{\text{odd}} - C)(x) \leq \beta|x|.$$
Linear gap and upper bound

**Theorem**

There exist a sequence $\omega$ such that for all $n$

\[
(C_{\text{odd}} + C_{\text{ev}} - C)(\omega_1 \ldots \omega_n) \geq n(\log \frac{4}{3})/2 - O(\log n).
\]

Moreover,

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C_{\text{odd}}(\omega_2\omega_1 \ldots \omega_{2n}\omega_{2n-1}) = C(\omega_1 \ldots \omega_{2n}) + O(\log n)
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\[
C_{\text{ev}}(\omega_2\omega_1 \ldots \omega_{2n}\omega_{2n-1}) \leq O(1).
\]

**Theorem**

There exist $\beta < \frac{1}{2}$ such that for large $x$

\[
(C_{\text{ev}} + C_{\text{odd}} - C)(x) \leq \beta |x|.
\]
Theorem

There exist a sequence $\omega$ such that for all $n$

$$(C_{\text{odd}} + C_{\text{ev}} - C)(\omega_1 \ldots \omega_n) \geq n(\log \frac{4}{3})/2 - O(\log n).$$

Moreover,

$$C_{\text{odd}}(\omega_2\omega_1 \ldots \omega_{2n}\omega_{2n-1}) = C(\omega_1 \ldots \omega_{2n}) + O(\log n)$$

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Outline

1. Introduction
2. Definitions and results
3. Application
4. Proof
Complex system
- two regions $\mathcal{X}$ and $\mathcal{Y}$ are interacting
- measurements in each region
  $$(x_1, y_1), (x_2, y_2), \ldots$$
- time resolution not enough to decide whether $x_i$ is a reply to $y_i$ or vice versa
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Decompressors are non-computable, in practice always computable (or extensible to comp ones)
- But, practical $\Rightarrow$ poly time computable $\Rightarrow$ asymmetry is restored, even for one pair of messages
  - assume factorizing is computationally difficult
  - $p, q$ be primes and $x_i = (p, q)$ and $y_i = pq$
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Main result: differ for some $(x, y)$ by $O(n)$ and factor $2 - \varepsilon$. 
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Information transfer (neurology and economics) measures influence of \( \mathcal{Y} \) on \( \mathcal{X} \) as

\[
H(X_n|X_{n-1}) - H(X_n|X_{n-1}, Y_{n-1})
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time delay for interaction assumed
without: \( H(X_n|X_{n-1}) - H(X_n|X_{n-1}, Y_n) \rightarrow \) symmetric
algorithmic information theory: \( IT(y \rightarrow x) = C(x) - C_{ev}(y_1 x_1 \cdots y_n x_n) \)
main result: for all \( \varepsilon > 0 \) there are \( x, y \) s.t. \( IT(y \rightarrow x) = C(x, y) + O(1) \) and
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example with primes \( \rightarrow \) good direction appears.

Unfortunately, in our example no direction of influence is natural.
Example where the direction means anything?
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The end, questions?
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Theorem

There exist a sequence $\omega$ such that for all $n$

$$(C_{\text{odd}} + C_{\text{ev}})(\omega_1 \cdots \omega_{2n}) \geq n \log \frac{4}{3} + C(\omega_1 \cdots \omega_{2n}) + O(\log n)$$

- State the problem in terms of on-line semimeasures,
- Game on strings of length 2,
- Concatenate the winning strategies.
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Probabilistic Turing machine $\leftrightarrow$ lower-semicomputable semimeasure

\[ U \]

\[ \begin{array}{ccccccc}
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & \ldots \\
\end{array} \]

\[ \leftarrow \]

\[ \begin{array}{ccccccc}
\ldots & & & & & & \\
\end{array} \]

\[ P : \{0, 1\}^* \rightarrow [0, 1] \text{ is a semimeasure if} \]

\[ P(x0) + P(x1) \leq P(x) \]

- There exist maximal lower-semicomputable semimeasures \( M(x) \).
- **Coding theorem:** \(- \log M(x) = C(x) + O(\log C(x)) \).
Probabilistic Turing machine $\leftrightarrow$ lower-semicomputable semimeasure

$P : \{0, 1\}^* \rightarrow [0, 1]$ is an even semimeasure if

$P(x_0) + P(x_1) \leq P(x)$ if $|x_0|$ is even,

$P(x_0) = P(x_1) = P(x)$ otherwise.

- There exist maximal lower-semicomputable even semimeasures $M_{ev}(x)$.
- Coding theorem[CSVV08]: $- \log M_{ev}(x) = C_{ev}(x) + O(\log |x|)$. 
(On-line) semimeasures and (on-line) coding theorem

Probabilistic Turing machine $\leftrightarrow$ lower-semicomputable semimeasure

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- There exist maximal lower-semicomputable even semimeasures $M_{ev}(x)$.
- **Coding theorem**\textbf{[CSVV08]}: $-\log M_{ev}(x) = C_{ev}(x) + O(\log |x|)$.

Warning: an even machine can not be modeled by products of l.s.c. $P_i$.

$P_{ev}(y_1 x_1 \cdots y_n x_n) = P_1(x_1|y_1) \cdot \ldots \cdot P_n(x_n|y_n) = P(x|y)$. 

Bruno Bauwens (Université de Lorraine, LORIA)
For all lsc $P_{odd}, P_{ev}$ there exist $\omega$ and lsc $P$ s.t. $(P_{odd} \cdot P_{ev})(\omega_1 \cdots \omega_{2n}) \leq \left(\frac{3}{4}\right)^n P(\omega_1 \cdots \omega_{2n})$
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Game for $|x| = 2$:
- Nature $\rightarrow P_{\text{odd}}, P_{\text{ev}}$
- Math $\rightarrow P$ s.t. $\sum_{|x|=2} P(x) \leq 3/4$

Math wins if either
- $P_{\text{odd}}(\varepsilon) > 1$
- $P_{\text{ev}}(\varepsilon) > 1$
- $P_{\text{ev}}(x) P_{\text{odd}}(x) \leq P(x)$
  for some 2-bit $x$
For all lsc $P_{\text{odd}}, P_{\text{ev}}$ there exist $\omega$ and lsc $P$ s.t. $(P_{\text{odd}} \cdot P_{\text{ev}})(\omega_1 \cdots \omega_{2n}) \leq \left(\frac{3}{4}\right)^n P(\omega_1 \cdots \omega_{2n})$

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For all lsc \( P_{\text{odd}}, P_{\text{ev}} \) there exist \( \omega \) and lsc \( \mathcal{P} \) s.t. \((P_{\text{odd}} \cdot P_{\text{ev}})(\omega_1 \cdots \omega_{2n}) \leq \left( \frac{3}{4} \right)^n P(\omega_1 \cdots \omega_{2n})\)

Game for \(|x| = 2\):
- Nature \( \rightarrow \) \( P_{\text{odd}}, P_{\text{ev}} \)
- Math \( \rightarrow \) \( \mathcal{P} \) s.t. \( \sum_{|x|=2} P(x) \leq \frac{3}{4} \)

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- \( P_{\text{ev}}(\varepsilon) > 1 \)
- \( P_{\text{ev}}(x)P_{\text{odd}}(x) \leq P(x) \)
  for some 2-bit \( x \)

Wait until either
- \( p > \frac{1}{2} \)
- \( u > \frac{1}{2} \)

Suppose \( p > \frac{1}{2} \)
For all lsc $P_{\text{odd}}, P_{\text{ev}}$ there exist $\omega$ and lsc $P$ s.t. $(P_{\text{odd}} \cdot P_{\text{ev}})(\omega_1 \cdots \omega_{2n}) \leq \left(\frac{3}{4}\right)^n P(\omega_1 \cdots \omega_{2n})$

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- Nature $\rightarrow P_{\text{odd}}, P_{\text{ev}}$
- Math $\rightarrow P$ s.t. $\sum_{|x|=2} P(x) \leq 3/4$
Math wins if either
- $P_{\text{odd}}(\varepsilon) > 1$
- $P_{\text{ev}}(\varepsilon) > 1$
- $P_{\text{ev}}(x)P_{\text{odd}}(x) \leq P(x)$
  for some 2-bit $x$

Suppose $p > \frac{1}{2}$
$G_n$ is played on restrict. $P_{\text{odd}}, P_{\text{ev}}$

How use output in $G_m$?

- Increases of $P$
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  - if all leafs are discarted then $P_{\text{odd}}(\epsilon) > 1$ or $P_{\text{ev}}(\epsilon) > 1$,
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