

# The Kolmogorov complexity of on-line predicting odd and even bits

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- 1 Introduction
- 2 Definitions and results
- 3 Application
- 4 Proof

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# Studying a theater play

play = 2 independent monologues  $x$ ,  $y$

- Someone studies  $x$  and  $y$ .
- Alice studies  $x$ , Bob studies  $y$ .

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Alice To be or not to be,

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Bob Once upon a time

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Splitting information “in pieces” does not increase the sum of parts of information.

$$H(X, Y) = H(X) + H(Y|X) \quad [= H(Y) + H(X|Y)]$$



# Shannon entropy: perfect symmetry of information

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Splitting in two parts increases the sum by at most  $O(\log |x|)$

$$C(x, y) = C(x) + C(y|x) + O(\log |x|).$$



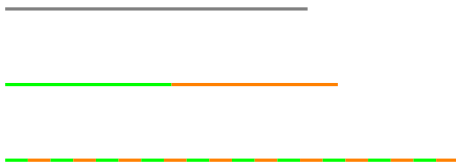
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- Main result:  $C_{\text{odd}}(x) + C_{\text{ev}}(x) \approx 2C(x)$  for infinitely many  $x$ .  
→ Confirming our example.

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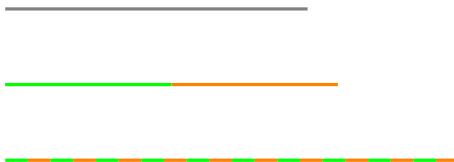
# Kolmogorov complexity: almost symmetry of information

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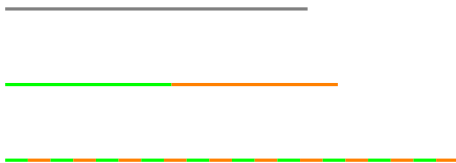
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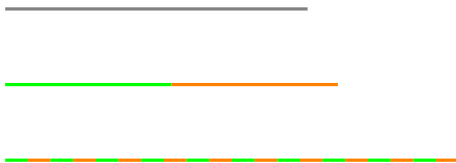
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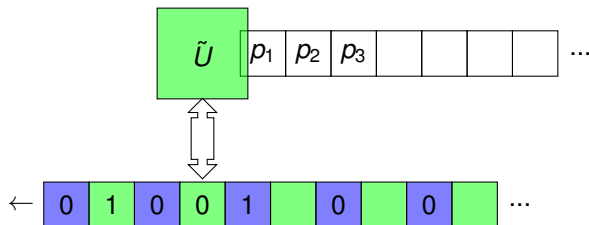
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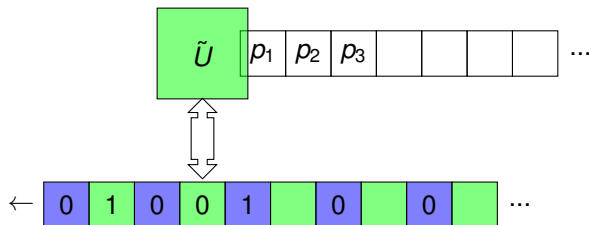
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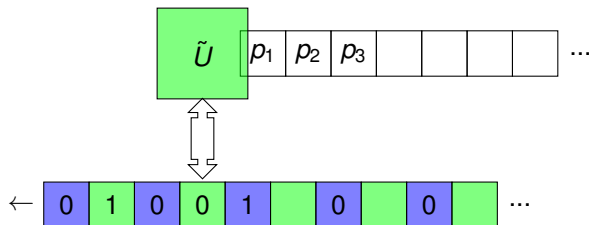
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**Even Kolmogorov complexity  $C_{\text{ev}}(x)$  is**

$$\min\{|p| : U(p, x_1 \cdots x_{i-1}) = x_i, i = 2, 4, \dots \leq |x|\}$$

Similar for  $C_{\text{odd}}$  [CSVV08].

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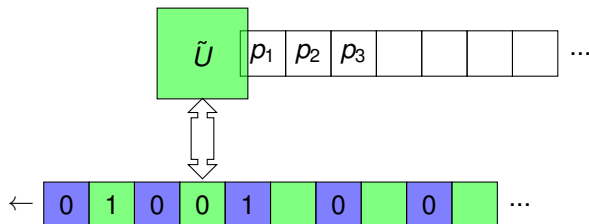
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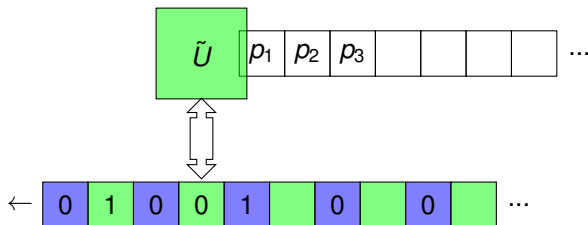
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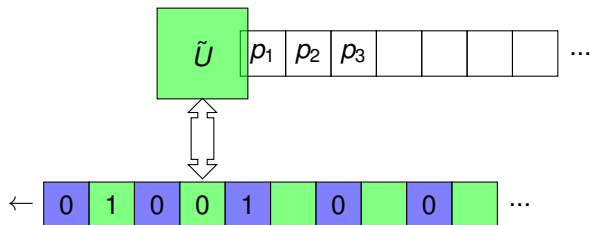
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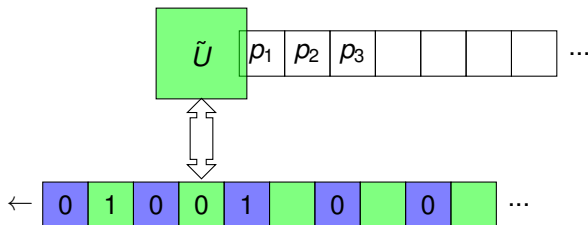
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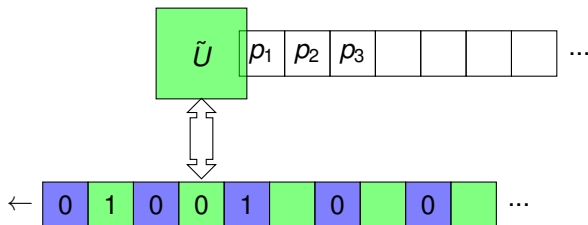
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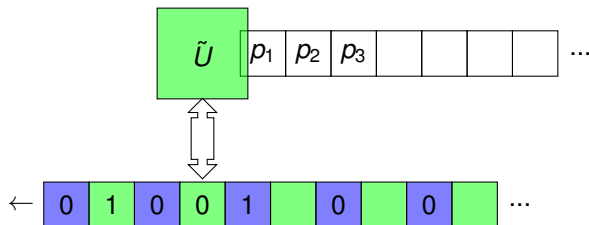
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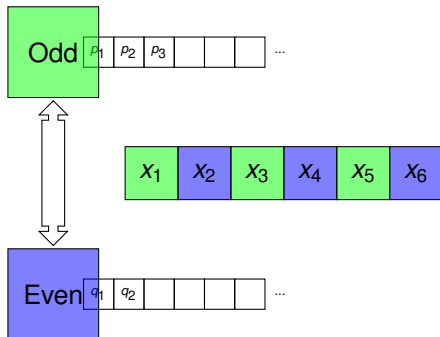
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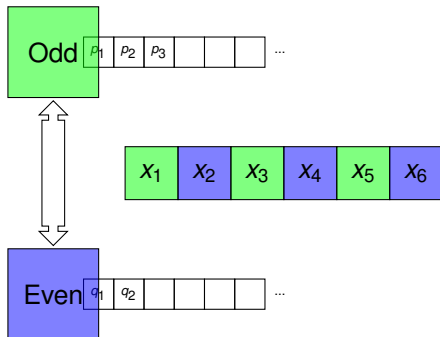
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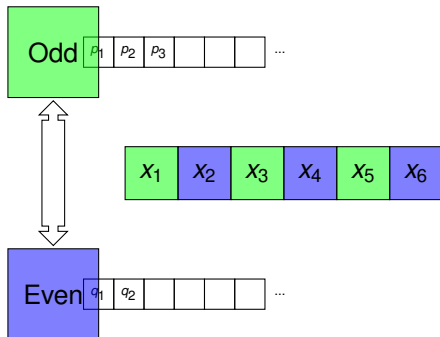
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Measure of (instantaneous) influence  
 $IT(y \rightarrow x) = C(x) - C_{\text{ev}}(y_1 x_1 \dots y_n x_n)$

$$IT(y \rightarrow x) = C(x, y) + O(1)$$

$$IT(x \rightarrow y) \leq \varepsilon C(x, y)$$



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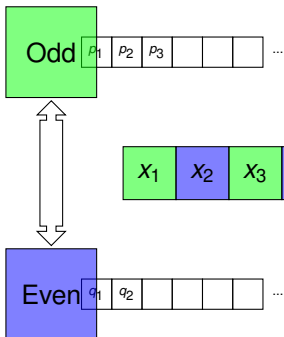
Measure of (instantaneous) influence  
 $IT(y \rightarrow x) = C(x) - C_{\text{ev}}(y_1 x_1 \dots y_n x_n)$

$$IT(y \rightarrow x) = C(x, y) + O(1)$$

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Asymmetry implies halting information

$$(C_{\text{odd}} + C_{\text{ev}})(x) - C(x) \leq (C - C^H)(x) + O(\log |x|).$$



## Theorem

For every  $\varepsilon > 0$  there exist  $\delta > 0$  and a sequence  $\omega$  such that for large  $n$

$$\frac{C_{\text{odd}}(\omega_1 \dots \omega_{2n})}{C_{\text{ev}}(\omega_1 \dots \omega_{2n})} \geq (1 - \varepsilon)C(\omega_1 \dots \omega_{2n}) + \delta n.$$

Moreover,

$$C_{\text{odd}}(\omega_2 \omega_1 \dots \omega_{2n} \omega_{2n-1}) = C(\omega_1 \dots \omega_{2n}) + O(\log n)$$

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## Theorem

For each  $k$  and  $\varepsilon > 0$  there exist  $\delta > 0$  and a sequence  $\omega$  such that for  $i \leq k$  and large  $n$

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There exist a sequence  $\omega$  such that for all  $n$

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Moreover,

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- 1 Introduction
- 2 Definitions and results
- 3 Application**
- 4 Proof



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- measurements in each region

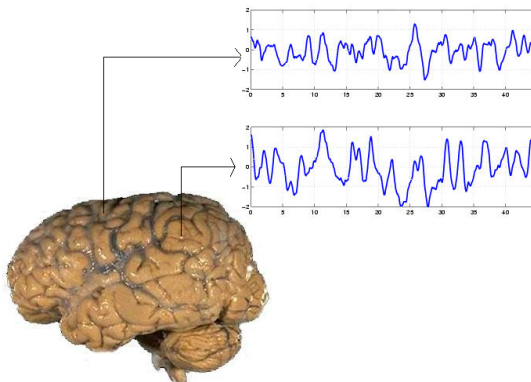
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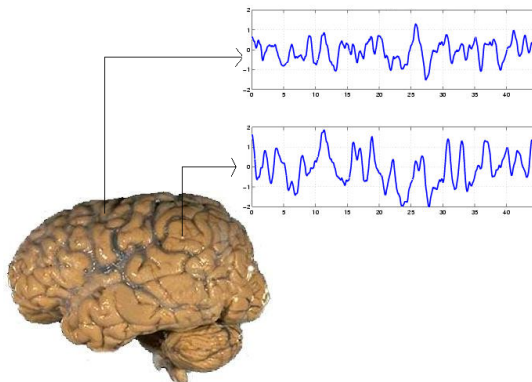
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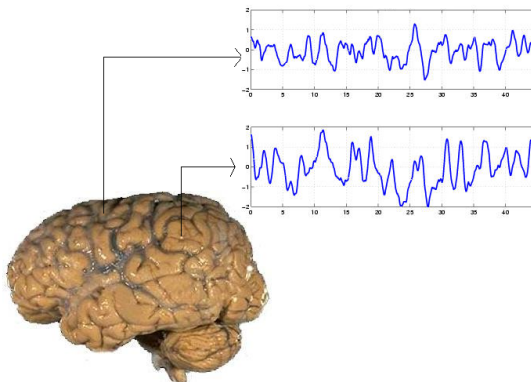
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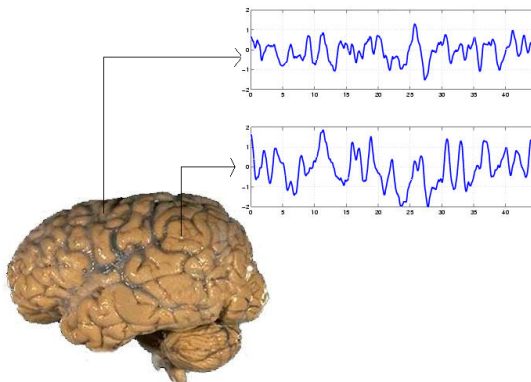
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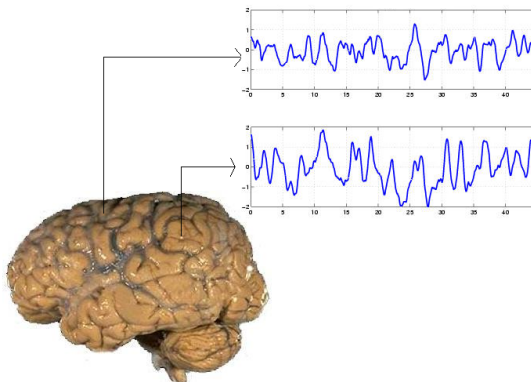
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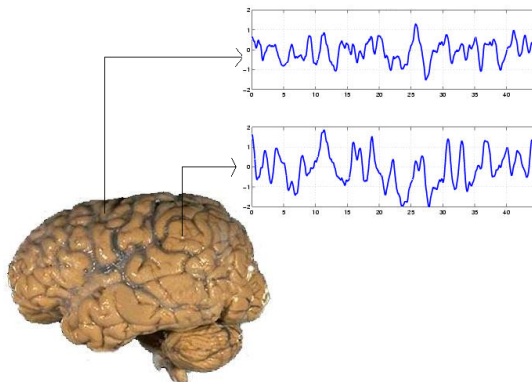
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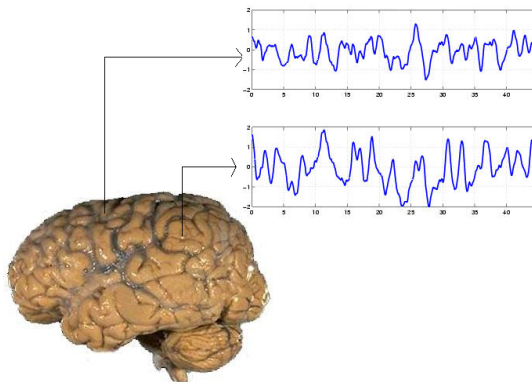
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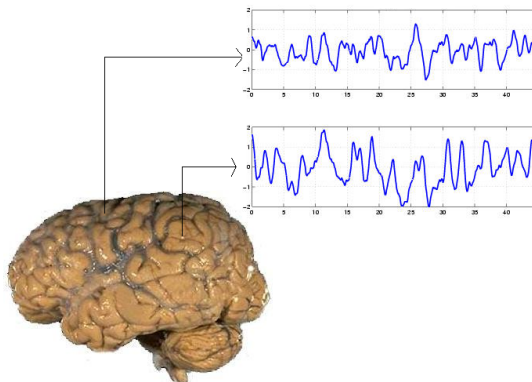
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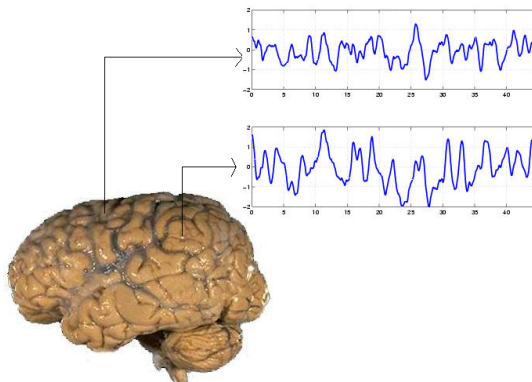
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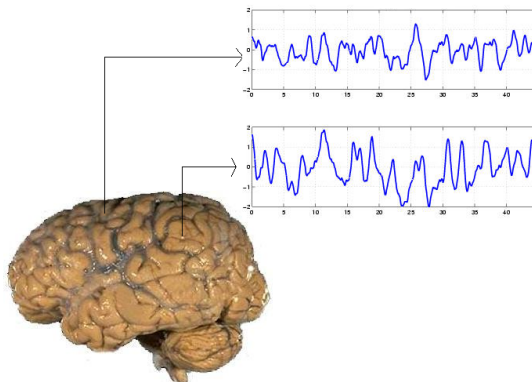
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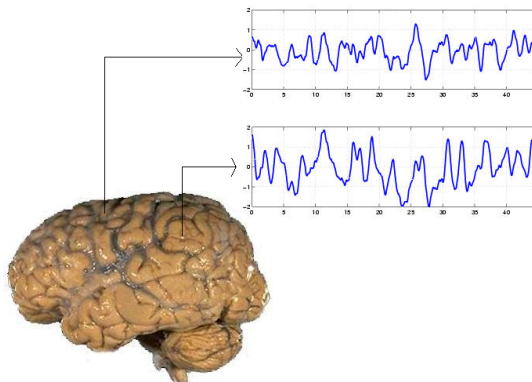
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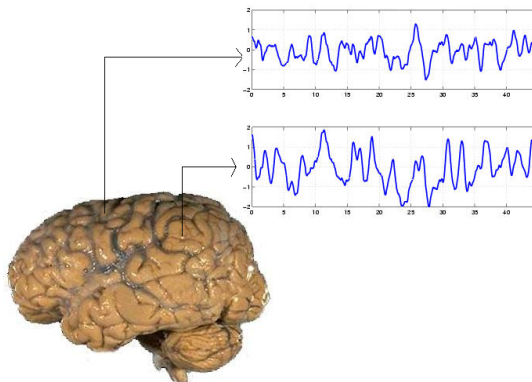
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- main result: for all  $\varepsilon > 0$  there are  $x, y$  s.t.  $IT(y \rightarrow x) = C(x, y) + O(1)$  and  $IT(x \rightarrow y) \leq \varepsilon C(x, y)$ ,
- example with primes  $\rightarrow$  good direction appears.

- Unfortunately, in our example no direction of influence is natural.  
Example where the direction means anything?

- information transfer (neurology and economics) measures influence of  $\mathcal{Y}$  on  $\mathcal{X}$  as

$$H(X_n|X_{n-1}) - H(X_n|X_{n-1}, Y_{n-1})$$

- time delay for interaction assumed
  - without:  $H(X_n|X_{n-1}) - H(X_n|X_{n-1}, Y_n) \rightarrow$  symmetric
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The end, questions?

- 1 Introduction
- 2 Definitions and results
- 3 Application
- 4 Proof**

## Theorem

*There exist a sequence  $\omega$  such that for all  $n$*

$$(C_{\text{odd}} + C_{\text{ev}})(\omega_1 \cdots \omega_{2n}) \geq n \log \frac{4}{3} + C(\omega_1 \cdots \omega_{2n}) + O(\log n)$$

- State the problem in terms of on-line semimeasures.
- Game on strings of length 2.
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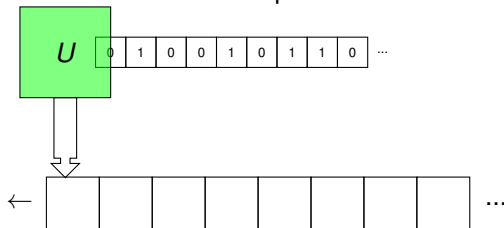
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Probabilistic Turing machine  $\leftrightarrow$  lower-semicomputable semimeasure

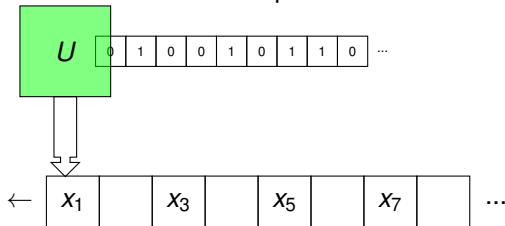


$P : \{0, 1\}^* \rightarrow [0, 1]$  is a semimeasure if

$$P(x0) + P(x1) \leq P(x)$$

- There exist maximal lower-semicomputable semimeasures  $M(x)$ .
- Coding theorem:  $-\log M(x) = C(x) + O(\log C(x))$ .

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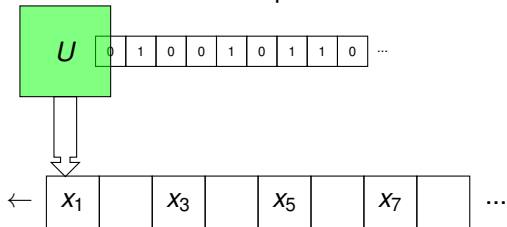


$P : \{0, 1\}^* \rightarrow [0, 1]$  is an **even** semimeasure if

$$\begin{aligned} P(x0) + P(x1) &\leq P(x) && \text{if } |x0| \text{ is even,} \\ P(x0) = P(x1) &= P(x) && \text{otherwise.} \end{aligned}$$

- There exist maximal lower-semicomputable even semimeasures  $M_{\text{ev}}(x)$ .
- **Coding theorem[CSVV08]:**  $-\log M_{\text{ev}}(x) = C_{\text{ev}}(x) + O(\log |x|)$ .

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Warning: an even machine can not be modeled by products of l.s.c.  $P_i$ .

$$P_{\text{ev}}(y_1 x_1 \cdots y_n x_n) = P_1(x_1 | y_1) \cdots P_n(x_n | y_n) = P(x | y).$$

For all lsc  $P_{\text{odd}}, P_{\text{ev}}$  there exist  $\omega$  and lsc  $P$  s.t.  $(P_{\text{odd}} \cdot P_{\text{ev}})(\omega_1 \cdots \omega_{2n}) \leq \left(\frac{3}{4}\right)^n P(\omega_1 \cdots \omega_{2n})$

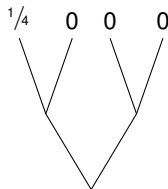
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Game for  $|x| = 2$ :

- Nature  $\rightarrow P_{\text{odd}}, P_{\text{ev}}$
- Math  $\rightarrow P$  s.t.  $\sum_{|x|=2} P(x) \leq 3/4$

Math wins if either

- $P_{\text{odd}}(\varepsilon) > 1$
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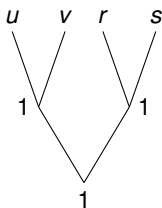
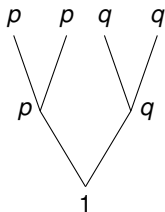
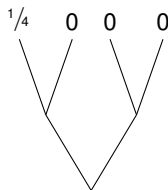
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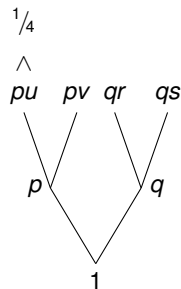
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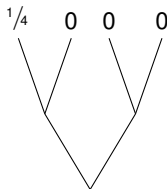
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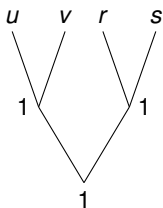
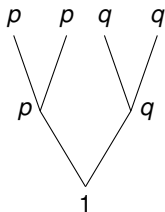
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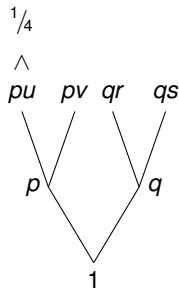
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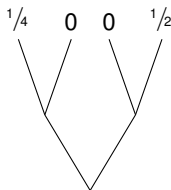
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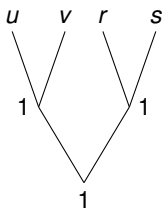
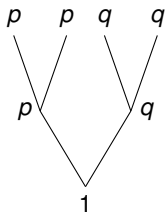
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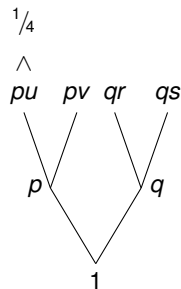
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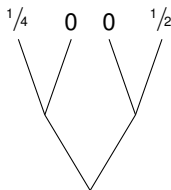
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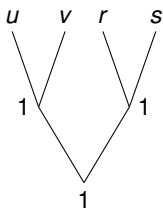
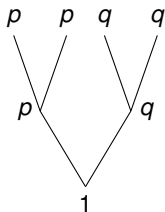
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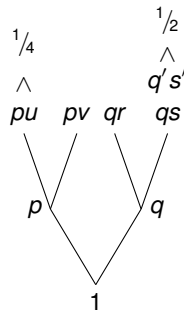
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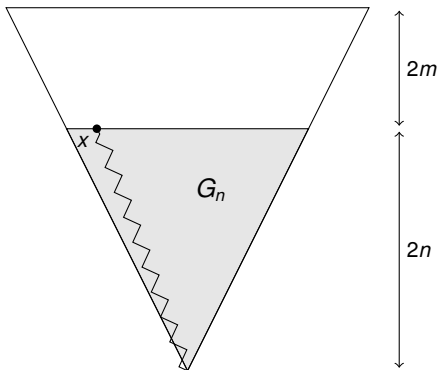


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# Concatinating strategies for $G_n$ and $G_m$ to $G_{n+m}$

$G_n$  is played on restrict.  $P_{\text{odd}}, P_{\text{ev}}$   
How use output in  $G_m$ ?

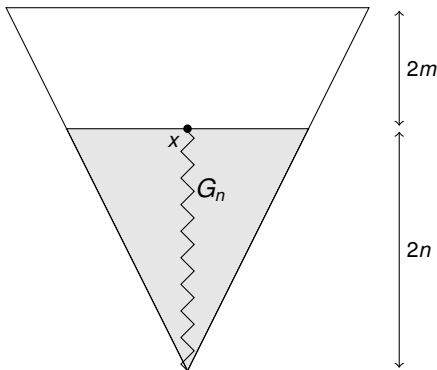


- Increases of  $P$  leaf by leaf,

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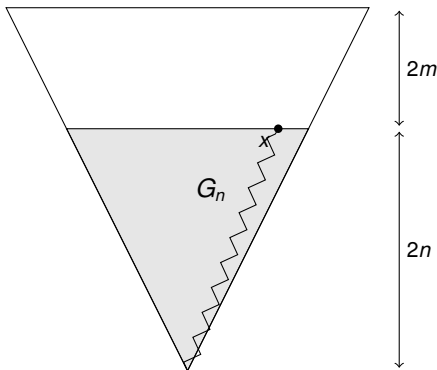


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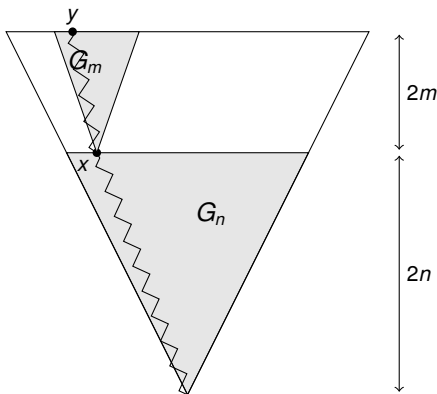


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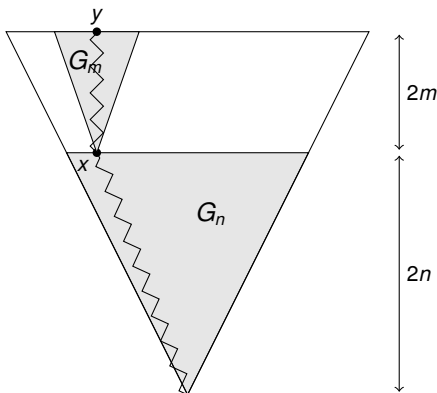


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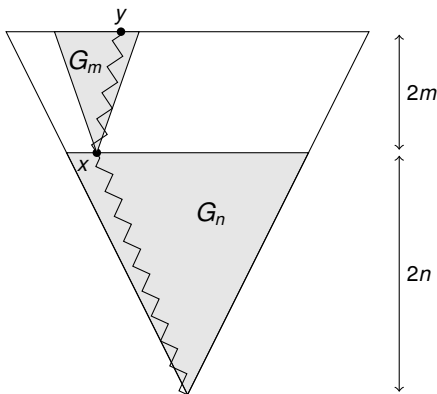


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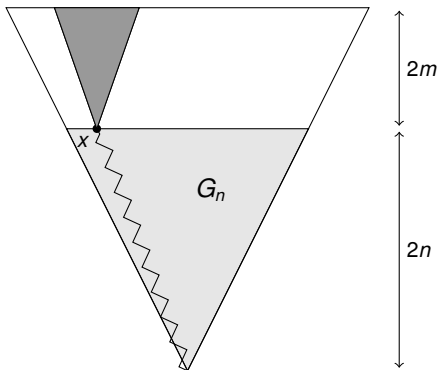
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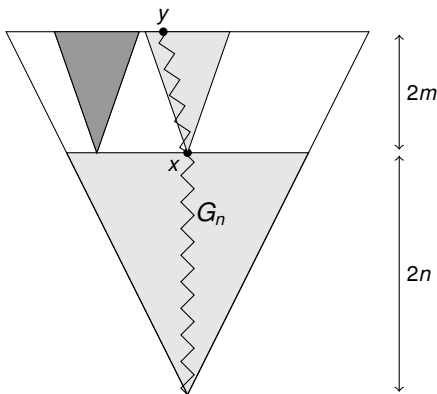


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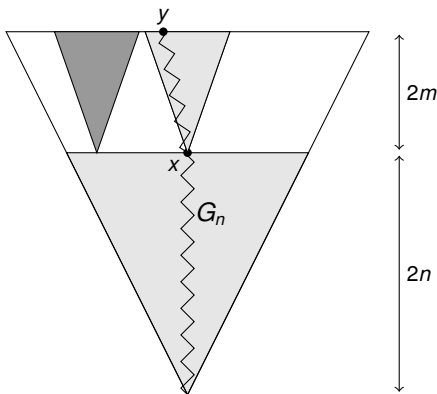


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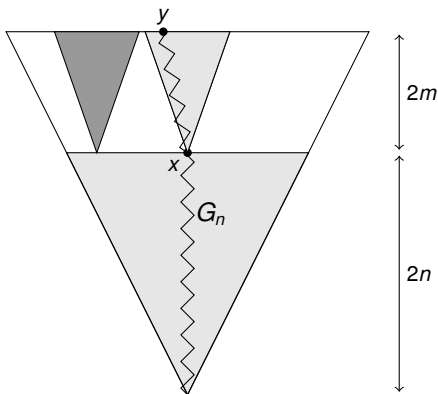
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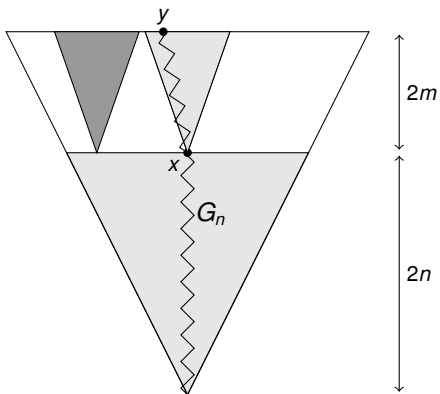
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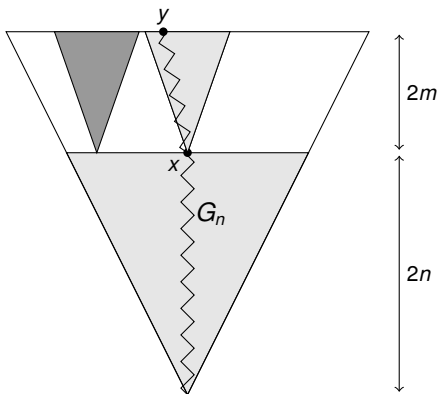
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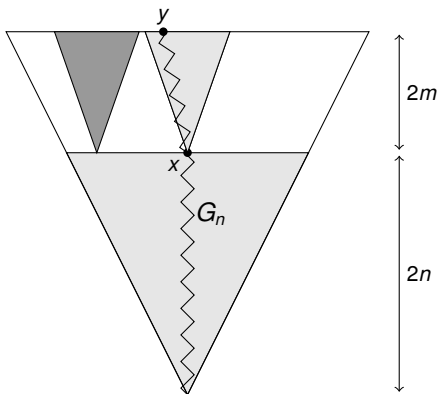
$G_n$  is played on restrict.  $P_{\text{odd}}, P_{\text{ev}}$   
How use output in  $G_m$ ?



- Increases of  $P$  leaf by leaf,
- $P(x) = o_x e_x$  product of upperbounds for  $P_{\text{odd}}(x), P_{\text{ev}}(x)$ ,
- discard leaf as one of the upperbounds is violated,
- if all leafs are discarded then  $P_{\text{odd}}(\varepsilon) > 1$  or  $P_{\text{ev}}(\varepsilon) > 1$ ,
- rescale inputs small game with  $o_x$  and  $e_x$ ,
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