Comparison of Garden Hose complexity with communication and circuit complexities

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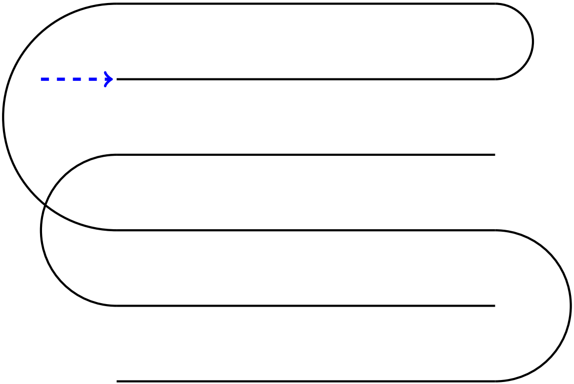
Garden Hose computation

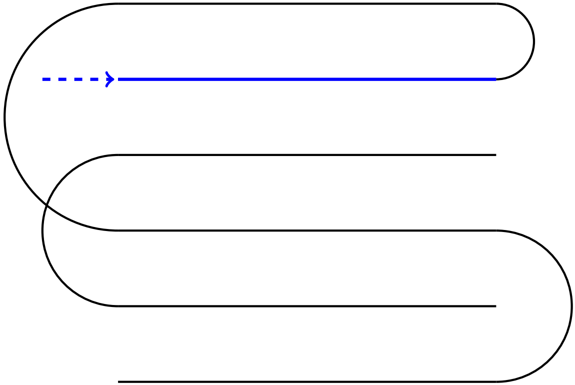
Introduced by Harry Buhrman, Serge Fehr, Christian Schaffner and Florian Speelman in «The Garden-Hose Model», 2011.

- Two participant: Alice and Bob.
- *k* parallel pipes between them.
- Alice and Bob connects some pairs of their ends with garden hose.
  Every end is either free or connected with exactly one other end on the same side.
- Alice connects water tap to one of her ends.
- Water goes from water tap through pipes and hose until it reaches free end.
- If this end is on Alice's side, computed value is 0, otherwise it is 1.

Garden Hose computation



Garden Hose computation



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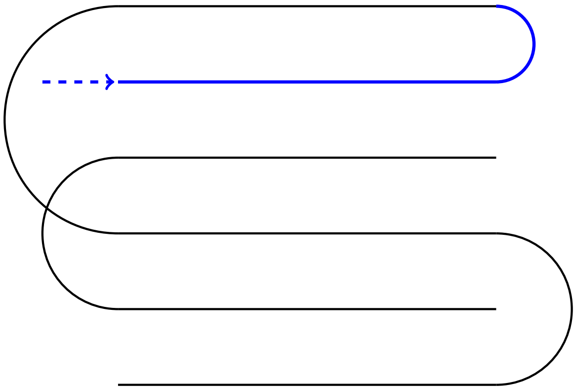

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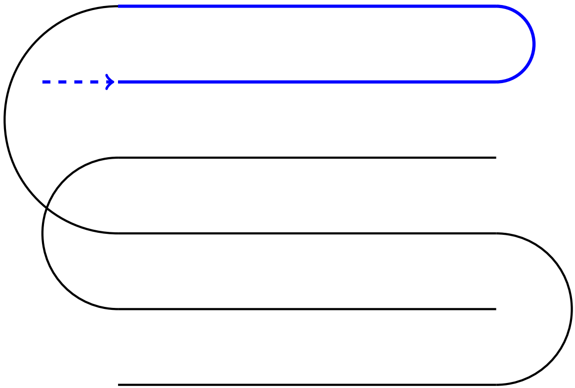

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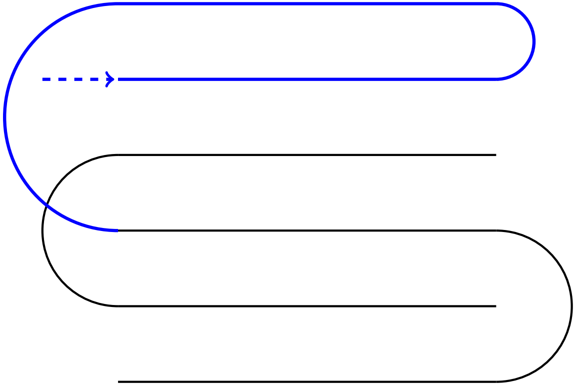

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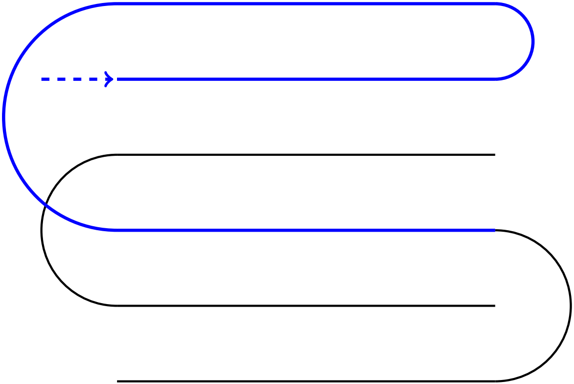

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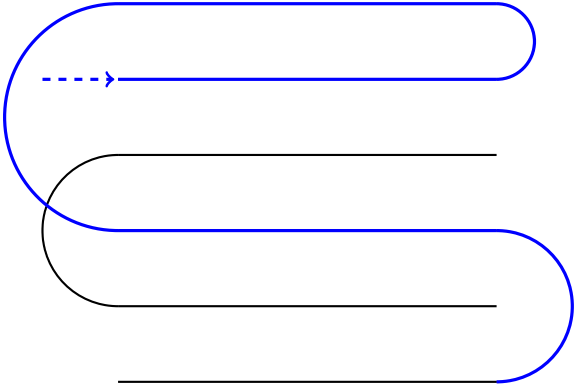

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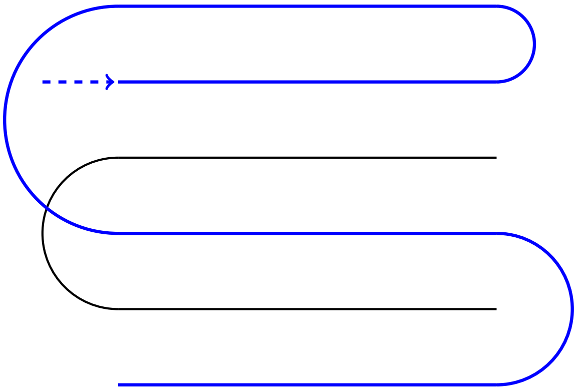

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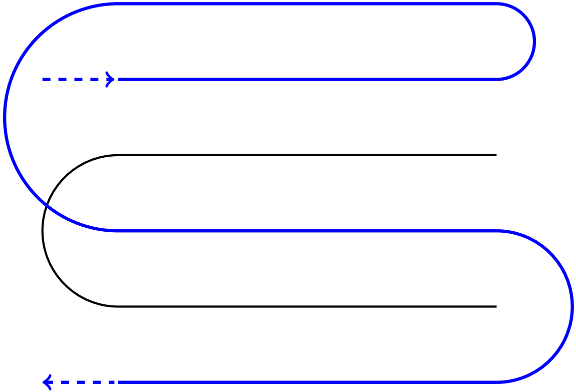

Definition of garden hose complexity

- We have boolean function $f\colon \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$
- Alice and Bob make their connections depending only on their part of input.
- Garden Hose complexity $GH(f)$ of f is minimal number of pipes for which it is possible to Alice and Bob make connections so for any input value of f will be computed correctly.

Known general bounds

- $GH(f) \leqslant 2^n + 1$
- $GH(f) \leqslant 2^{CC(f)+1}$
- $GH(f) \log(GH(f)) \geqslant CC(f)$
- There exist function f for which $GH(f)$ is exponential

Known bounds for specific functions

- $n \leqslant GH(EQ_n) \leqslant 1.5n$
- $\frac{n}{\log n} \leqslant GH(IP_n), GH(GT_n), GH(MAJ_n)$
- $GH(IP_n) \leqslant 4n + 1$
- $GH(GT_n) \leqslant 5n$
- $GH(MAJ_n) \leqslant (n + 2)^2$

Explicit function which GH is exponential with CC

- $n = 2^k, f\colon \{0,1\}^k \times \{0,1\}^n (= 2^{\{0,1\}^k}) \to \{0,1\}$
  $f(x,y) = 1 \Leftrightarrow x \in y$
- $CC(f) \leqslant k+1$: Alice sends x to Bob, and he answers if $x \in y$
- For any different y_1 and y_2 Bob has to make different connections.
- There are less then $m!$ different connections for m pipes.
- $GH(f)! \geqslant 2^n$
  $GH(f) = \Omega\left(\frac{2^n}{n}\right)$
  $GH(f) = \Omega(2^{2^k - k})$

Comparison with circuit complexity

- Known: if f can be computed with circuit of depth k, then $GH(f) = O(4^k)$.
- If $f\colon \{0,1\}^k \times \{0,1\}^k \to \{0,1\}$ and $GH(f) \leqslant n$, then there are functions
  $\alpha, \beta : \{0,1\}^k \to \{0,1\}^{n^2}$
  $\gamma : \{0,1\}^n \to \{0,1\}^n$
  $g : \{0,1\}^{n^2} \times \{0,1\}^{n^2} \times \{0,1\}^n \to \{0,1\}$
  such that $f(x,y) = g(\alpha(x), \beta(y), \gamma(x))$ and g can be computed with scheme of depth $O(\log^2(n))$.
- Local preprocessing is necessary.

Local preprocessing

- Let us think about α and β values as matrices $n \times n$.
- $\alpha(x)_{i,j} = 1$ if and only if Alice connects pipes i and j on input x, and similarly for β and Bob.
- $\gamma(x)_i = 1$ if and only if Alice connects water tap to ith pipe.

Intermediate values

- Now we introduce pass-through numeration of all ends: $0, 2, \dots$ for Alice's ends and $1, 3, \dots$ for Bob's. Ends of pipe number k are $2k$ and $2k+1$.
- Function: $s_k(i, j) = 1$ if and only if water can get from end i to end j in at most 2^k steps (one step is pass through one pipe or piece of hose).
- $s_0(i, j) = 1$ in the following cases:
  - $i = j$
  - $\{i, j\} = \{2c, 2c+1\}$ for some c
  - $\{i, j\} = \{2c, 2d\}$ for some c and d and $\alpha_{c,d} = 1$
  - $\{i, j\} = \{2c+1, 2d+1\}$ for some c and d and $\beta_{c,d} = 1$

  Otherwise, $s_0(i, j) = 0$.

Computing final value

- $s_{k+1}(i,j) = \bigvee_t s_k(i,t) \wedge s_k(t,j)$
- All s_{k+1} can be computed by circuit of depth $O(\log(n))$ in parallel.
- GH computes value equal to $\bigvee_{t=0}^{n} \bigvee_{u=0}^{n} \gamma_t \wedge s_{\lceil \log(n) \rceil + 1}(t, 2u+1) \wedge \neg \bigvee_{v=0}^{n} \beta_{u,v}$
- So, given values of α, β, γ GH computations can be emulated with circuit of depth $(\lceil \log(n) \rceil + 1) \cdot O(\log(n)) = O(\log^2(n))$

Open questions

- Explicit function with overlinear garden hose complexity.
- Explicit function f such that $CC(f)$ is linear, and $GH(f)$ is exponential.
- Does there exist function f such that $CC(f) > GH(f)$?
- (special case) What is communication complexity of function f (is it greater then n?): Alice and Bob get permutations α and β on n elements, and f is equal to 1 if and only if 0 is in the odd length cycle in permutation $\alpha\beta$?

End

Thanks for listening!
Questions?