Higher randomness

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Higher randomness

Section 1

Introduction

What are Π^1_1-sets ?

A good intuitive way to think of Δ^1_1 and Π^1_1 sets :

Theorem (Hamkins, Lewis)

The Δ^1_1 sets of integers are exactly those that can be decided in a computable ordinal length of time by an infinite time Turing machine.

An extension of the theorem :

The Π^1_1 sets of integer are exactly those one can enumerate in a computable ordinal length of time by an infinite time Turing machine.

Motivation

- A very rich theory of computable randomness has been developed during the last twenty years.
- A very rich theory of Higher computability has been developed, lying between computability and effective descriptive set theory.
- Time to mix them !

What part of this theory works in the Higher world ?

The Higher world

Here are the obvious higher analogue in the of usual notions in the bottom world.

The bottom world	**The higher world**
finite time t	computable ordinal time α
computable $\leftrightarrow \Delta^0_1$	Δ^1_1
c.e. $\leftrightarrow \Sigma^0_1$	Π^1_1
$A \geqslant_T X \leftrightarrow X$ is $\Delta^0_1(A)$	$A \geqslant_h X \leftrightarrow X$ is $\Delta^1_1(A)$

Forcing continuity

Unlike in the "bottom" world, where a Turing reduction is coutinuous, an *h*-reduction can require infinitely many bits of the input to decide only finitely many bits of the output. It's a problem to "import" results of the bottom world into the higher world. As an example :

The higher world	The bottom world
Any Π^1_1 set which is not Δ^1_1 can *h*-compute any Π^1_1 set	Any *c.e.* set which is not computable can Turing compute any *c.e.* set ? ? ?

One main reason for this is that Π^1_1 sets which are not Δ^1_1 increase ω_1^{ck}, the smallest non-computable ordinal.

One solution : **Forcing continuity.**

The first Δ^1_1 continuous reduction

The first attempt to use continuous version of hyperarithmetic reducibility was made by Hjorth and Nies in order to study higher analogue of Kucera-Gacs and Higher analogue of Base for randomness.

Definition

A *fin-h* reduction is a partial Π^1_1 map $M \subseteq 2^{<\omega} \times 2^{<\omega}$ which is :

- Consistent : If τ_1 is mapped to $\sigma^\frown 0$ and τ_2 is mapped to $\sigma^\frown 1$ then we must have $\tau_1 \perp \tau_2$
- Closed under prefixes : If τ is mapped to something, any prefix of τ should be mapped to something.

We say that $A \geqslant_{fin-h} X$ if for a *fin-h* reduction M we have $\forall n \ \ \exists \tau \prec X \ \ \exists \sigma \prec A \ \ |\tau| \geqslant n \wedge \langle \sigma, \tau \rangle \in M$.

What happened ?

What are the properties of *fin-h* reductions ? We have three things to say which will each initiate three different parts of the talk :

- One good news
- One bad news
- One surprise

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The good news !

The Higher Kucera-Gacs works with continuous reduction. Great ! Of course it also works with hyperarithmetic reduction... But the computation can even be made effectively continuous.

This comes next to a theorem of Martin and Friedman, saying that an uncountable closed Σ^1_1 class contains members above any hyperarithmetical degree. So Higher Kucera-Gacs says that if the class have positive measure, then the computation can be made continuous.

The bad news

Base for randomness does not work as expected. The higher version of this notion is equivalent to Δ^1_1. The reason is that :

- Continous Turing reduction is used to compute the oracle
  but
- Full power of the oracle is used for relativization.

We need to investigate what could be a "continuous way" to use the oracle.

The surprise

The reduction itself defined by Hjorth and Nies seems perfectible.

Sometimes...

Sometimes everything works exactly the same way in the bottom world and in the Higher world.

But...

But there are also things which work differently and it took us time to identify all the traps in which not to fall !

Higher randomness

Section 2

Higher continuous reductions

The first Δ^1_1 continuous reduction

In the bottom world, the following four definitions are equivalent :

1. $A \geqslant_T X$
2. There is a Σ^0_1 partial map $R : 2^{<\omega} \to 2^{<\omega}$, **consistent on prefixes of** A, such that
$\forall n \;\; \exists \tau \prec X \;\; \exists \sigma \prec A \;\; |\tau| \geqslant n \wedge \langle \sigma, \tau \rangle \in R$
3. There is a Σ^0_1 partial map $R : 2^{<\omega} \to 2^{<\omega}$, **consistent everywhere**, such that
$\forall n \;\; \exists \tau \prec X \;\; \exists \sigma \prec A \;\; |\tau| \geqslant n \wedge \langle \sigma, \tau \rangle \in R$
4. There is a Σ^0_1 partial map $R : 2^{<\omega} \to 2^{<\omega}$, **consistent everywhere and closed under prefixes**, such that
$\forall n \;\; \exists \tau \prec X \;\; \exists \sigma \prec A \;\; |\tau| \geqslant n \wedge \langle \sigma, \tau \rangle \in R$

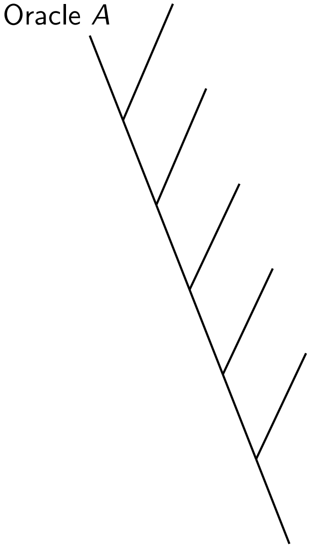
The reduction *fin-h* defined at first By Hjorth and Nies is exactly this last definition when we replace Σ_1^0 by Π_1^1.

A topological difference

The bottom world	The higher world
At any time t of the enumeration, the set of strings mapped so far is a **clopen set**	At any time α of the enumeration, the set of strings mapped so far is an **open set**.

This make the three previous notions different in the higher world.

The Fishbone



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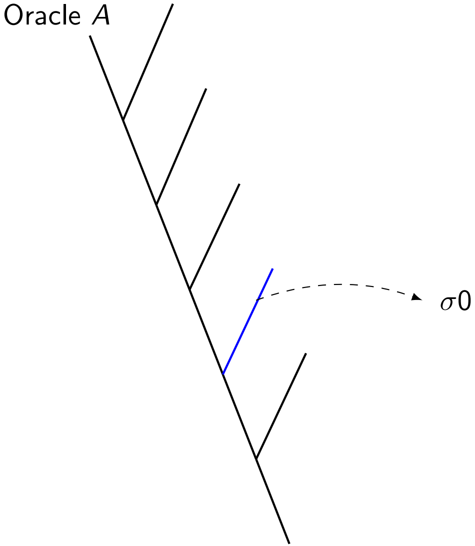

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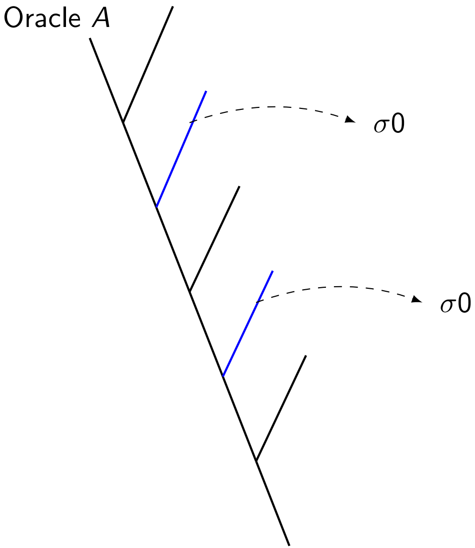

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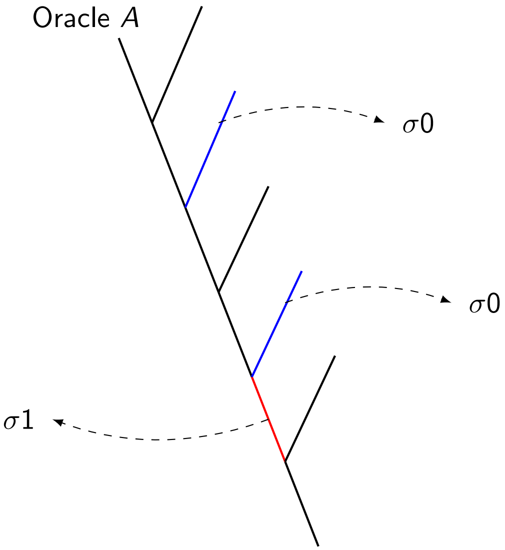

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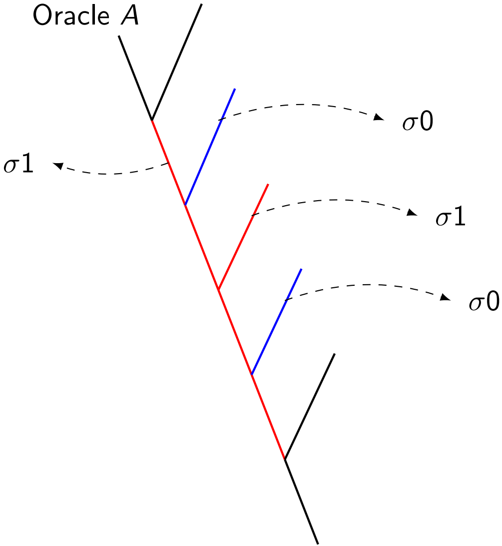
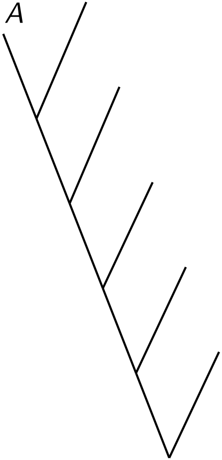


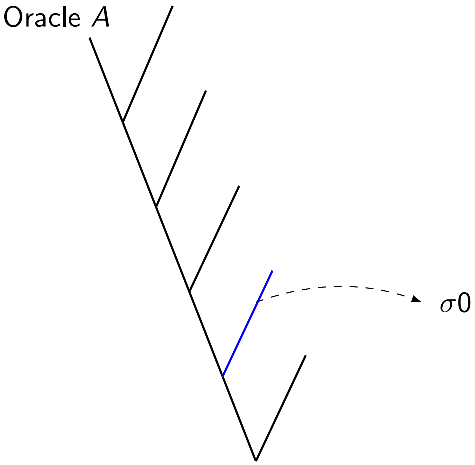
Defeating fin-h

Oracle A



Basic strategy :

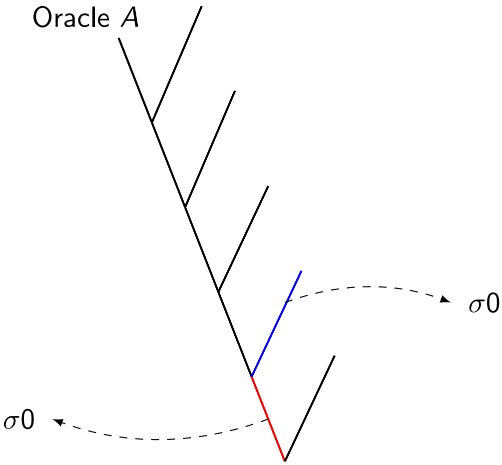
Defeating fin-h



Basic strategy :

Wait for
the opponent to decide
sth. on all the prefixes.

Defeating fin-h



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Suppose it matches one
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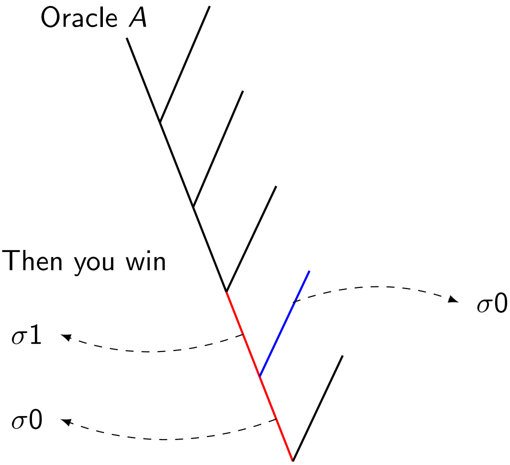

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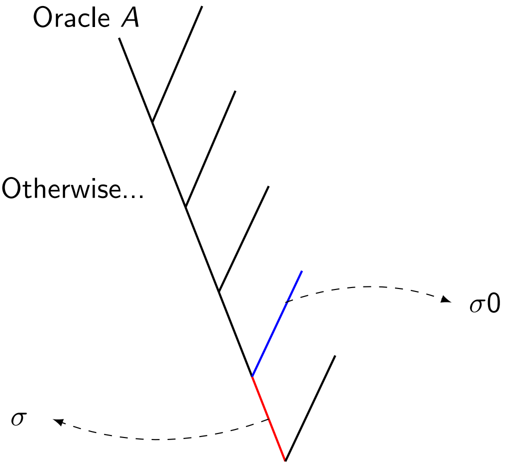

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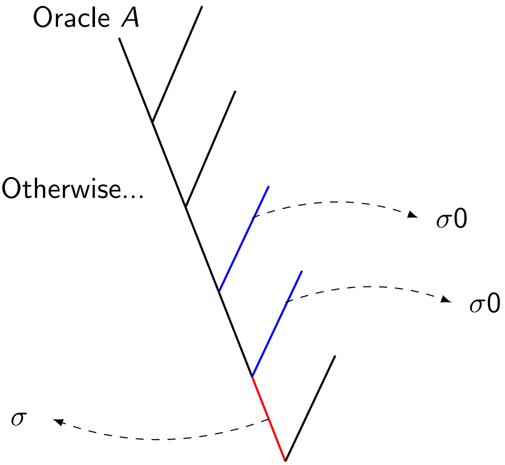

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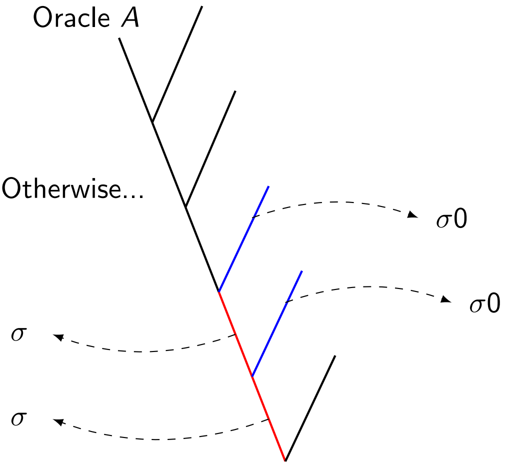
Defeating fin-h



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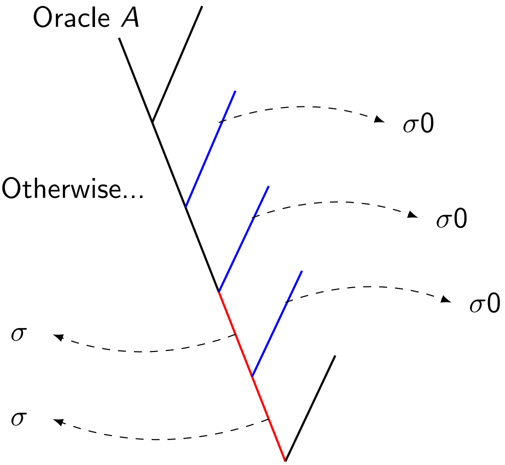

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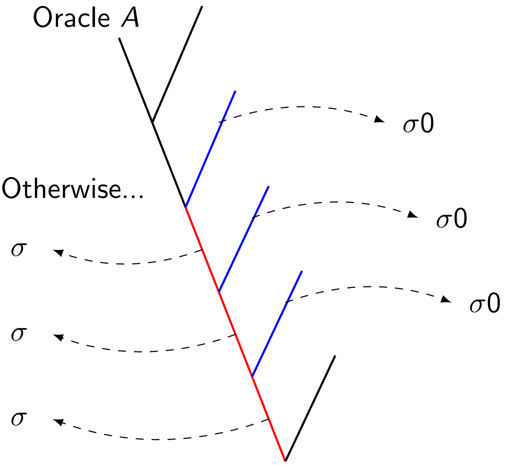

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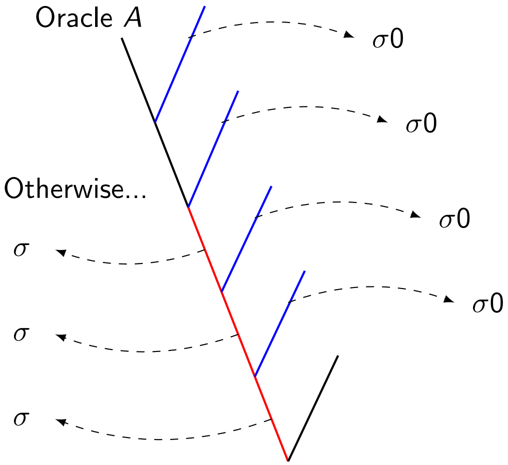

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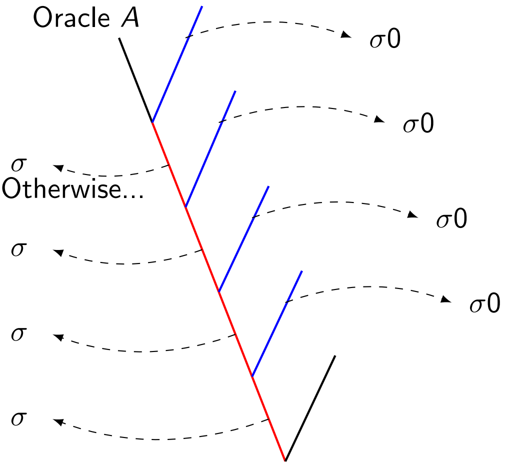

Basic strategy :

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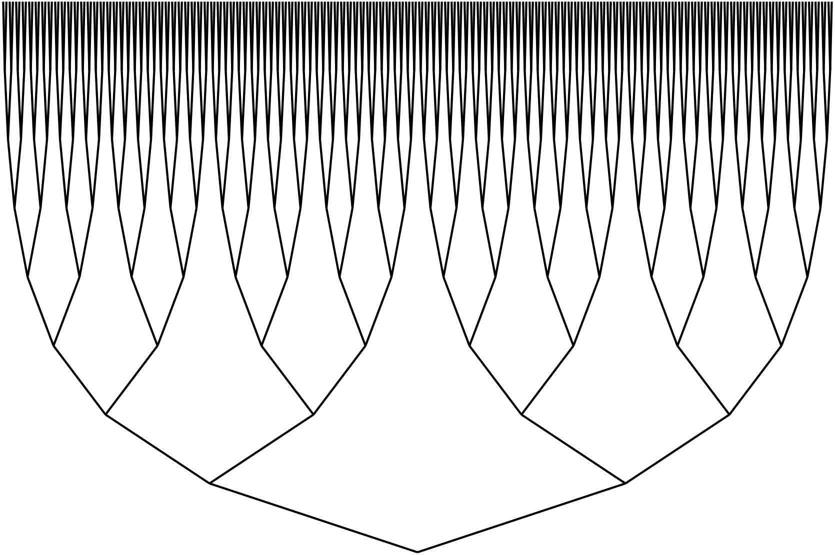
Defeating fin-h

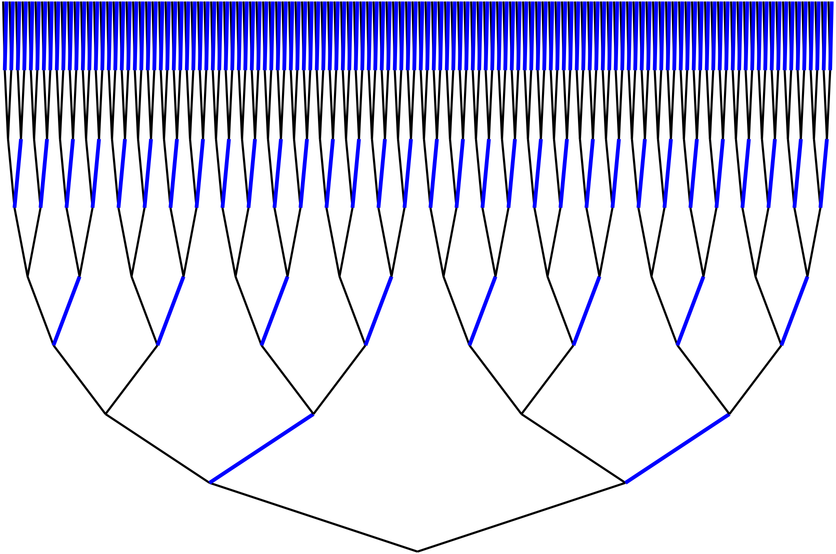
This is only one strategy. The problem is that one machine can force you to pick an entire oracle in order to defeat it. How to continue the construction and defeat other requirements ?

One solution : The perfect treesh-bone !

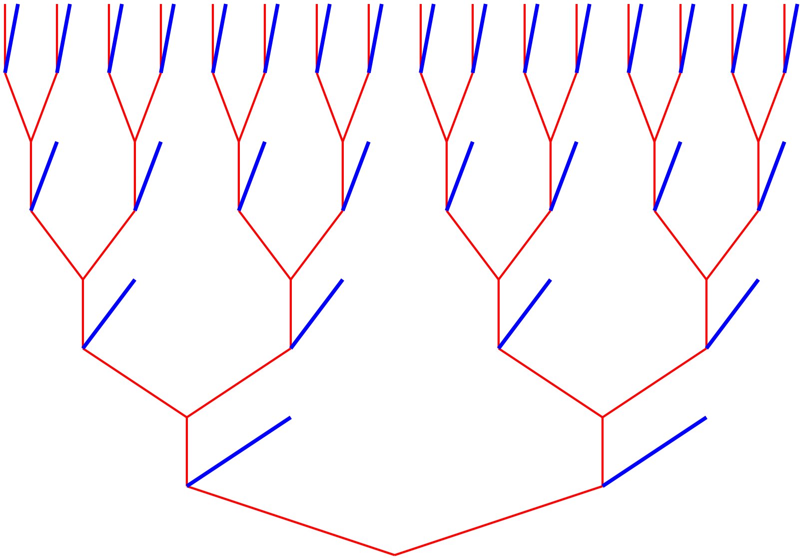
The treesh-bone (1)



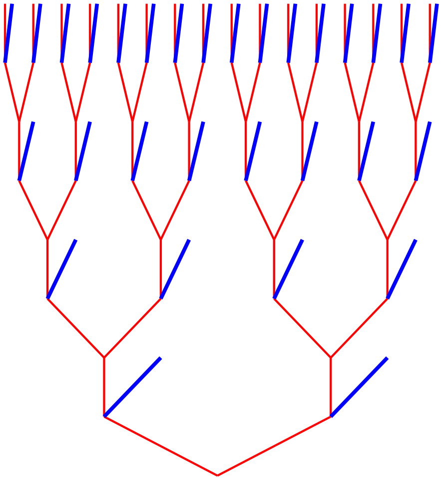
The treesh-bone (2)



The treesh-bone (3)



The treesh-bone (4)



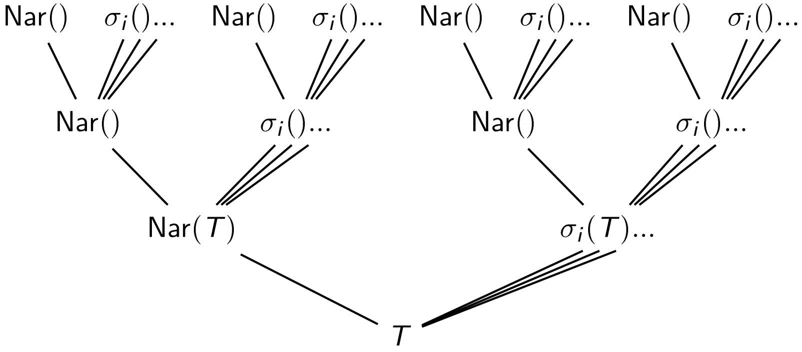
Put $\sigma 0$
along all the blue strings

Even if we
are forced to stay along the red part of the tree, we still have a prefect tree that we can continue to work with !

— :Nar(T),
The narrow subtree of T
— :$\sigma_i(T)$, the subtree of T extending the string σ_i

The tree of trees (1)

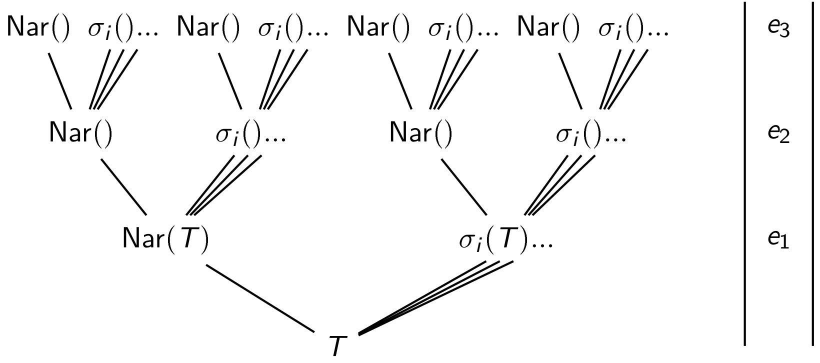
We can imagine that working in a tree of trees.



- The left node of T correspond to $Nar(T)$
- There is infinitely many right node $\sigma_i(T)$

The tree of trees (2)

We now order the requirement to do a higher finite injury argument :



The Higher Turing reduction

So some consistent map of strings cannot be made equivalent to some consistent map of strings whose domain is closed by prefixes.

Similarly we can prove that if a map of strings, not consistent everywhere, sends X to Y, there is not necessarily a consistent map of strings sending X to Y.

These brings the new definition :

Definition

We say that $A \geqslant_T B$ if there is a Π^1_1 partial map $R : 2^{<\omega} \to 2^{<\omega}$, **consistent on prefixes of** A, such that $\forall n \, \exists \tau \prec X \;\; \exists \sigma \prec A \;\; |\tau| \geqslant n \wedge \langle \sigma, \tau \rangle \in R$.

Oracles for which reductions collapses

For a large class of oracles, in a measure theoretic sense, the three notions of reductions are the same :

Fact

If $\omega_1^A = \omega_1^{\text{ck}}$ and $A \geqslant_T X$ then $A \geqslant_{fin-h} X$.

Higher randomness

Section 3

Higher continuous relativization

Relative randomness

As for Turing reduction, the most immediate way to think the higher analogue of Martin-Löf random relatively to some oracle A is to use the full power of A :

The bottom world	The higher world
The class U_n is $\Sigma_1^0(A)$	The class U_n is $\Pi_1^1(A)$

But this is giving too much power to A.

Relative randomness

We introduce continuous relativization :

Definition

A A-Π^1_1 Martin-Löf test is a subset M of $2^{<\omega} \times 2^{<\omega} \times \omega$ such that for evey n the open set $\{[\tau] \mid \exists \sigma \prec A \ \ (\sigma, \tau, n) \in M\}$ has measure smaller than 2^{-n}.

Again, this notion is inspired by some equivalences that we can find in the bottom world.

Uniform Test

In the bottom world, we have a trimming lemma :

Definition

We can uniformily transform a Σ^0_1 subset M of $2^{<\omega} \times 2^{<\omega}$ into another set $\tilde{M}$ such that

- $\forall X$ the open set $\{[\tau] \mid \exists \sigma \prec X \ \ (\sigma, \tau, n) \in \tilde{M}\}$ has measure smaller than 2^{-n}
- $\forall X$ if the open set $\{[\tau] \mid \exists \sigma \prec X \ \ (\sigma, \tau, n) \in M\}$ has already measure smaller than 2^{-n} then it remains unchanged in $\tilde{M}$.

This leads to the fact that there is a universal Martin-Löf Test, uniformly in every oracle.

No Universal Uniform Martin-Löf Test

But for the same reason as with the Turing reduction, the triming lemma does not seems to work. In fact we will now prove the following theorem :

Theorem (BGM)

There exists an oracle A such that for any "oracle open set" U_e, if $\forall X \quad U_e^X \neq 2^\omega$ then there exists Y_e such that

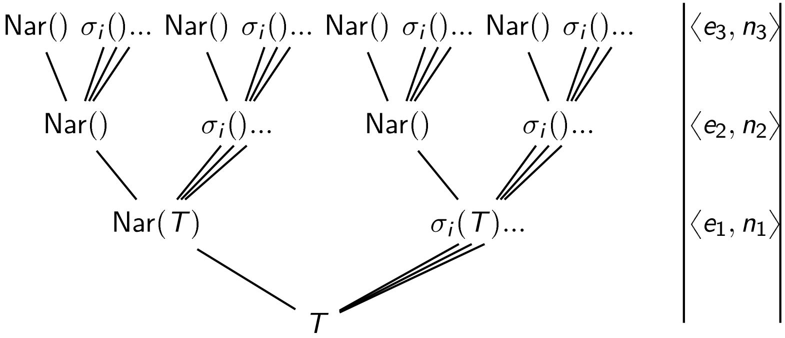
- $Y_e \notin U_e^A$
- $A \geq_T Y_e$

Corrolary (BGM)

For some oracle A, there is no A-universal uniform Martin-Löf test. (Since we cannot even get the first component of the test right...)

No Universal Uniform Martin-Löf Test

At level $h = \langle e_i, n_i \rangle$ of the tree of trees we ensure that if $U_e^A \neq 2^\omega$ then the n first bits of Y_i does not belongs to U_e^A.



Also at level $h = \langle e_i, n_i \rangle$ we continue the enumeration of the reduction of X_e to A so that A computes the n first bits of X_e.

No Universal Martin-Löf Test

But we also have a stronger result :

Theorem (BGM)

There exists an oracle A such that for any "oracle open set" U_e, if $U_e^A \neq 2^\omega$ then there exists Y_e such that

- $Y_e \notin U_e^A$
- Y_e is not A-MLR (Y_e belongs to another A-test $\bigcap_n V_{e,n}^A$)

Corollary (BGM)

For some oracle A, there is no A-universal Martin-Löf test.

However...

One might be disappointed by a notion of relativization which does not work. However a universal test still exists for a large class of oracles :

Theorem (BGM)

If A is higher MLR or higher 1-generic then there is a A-universal Martin-Löf test (but not necessarily uniform)

Theorem (BGM)

If A is higher-tt below Klenee's $\mathcal{O}$ then there is a A-universal uniform Martin-Löf test

Thank you. Questions ?