Closed Left-R.E. Sets

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1. R.e. and left-r.e. sets

A nonempty set $\mathbf{A}$ is r.e. iff it is the range of a recursive function.

Alternatively, one can say $\mathbf{A}$ is r.e. iff it is the limit of a uniformly recursive sequence $\mathbf{A_0, A_1, \ldots}$ of sets with $\mathbf{A_n \subseteq A_{n+1}}$ for all $\mathbf{n}$.

This can be generalised to approximation from the left. Let $\mathbf{A_n \leq_{lex} A_{n+1}}$ if either $\mathbf{A_n = A_{n+1}}$ or the minimal element $\mathbf{x}$ of the symmetric difference of $\mathbf{A_n}$ and $\mathbf{A_{n+1}}$ is in $\mathbf{A_{n+1}}$.

A set $\mathbf{A}$ is left-r.e. iff it is the limit of a uniformly recursive sequence $\mathbf{A_0, A_1, \ldots}$ of sets with $\mathbf{A_n \leq_{lex} A_{n+1}}$ for all $\mathbf{n}$.

R.e. sets are left-r.e. sets.
Reason: If $\mathbf{A_n \subset A_{n+1}}$ then $\mathbf{A_n <_{lex} A_{n+1}}$.

Examples

The halting problem $\mathbf{K}$ is the most famous example of an r.e. set which is not recursive.

Every set $\mathbf{A}$ has a real value $\mathbf{real(A)} = \sum_{\mathbf{n} \in \mathbf{A}} \mathbf{2^{-n-1}}$ and $\mathbf{A}$ is left-r.e. iff $\mathbf{real(A)}$ is the supremum of a recursive sequence of rationals.

Chaitin [1975] gave the example of a set, called $\mathbf{\Omega}$, which is left-r.e. but not r.e.: Here the idea is to construct a universal prefix-free Turing machine $\mathbf{TM}$ which reads bits from a tape and calculates in parallel; the set $\mathbf{\Omega}$ is chosen such that $\mathbf{TM}$ halts with probability $\mathbf{real(\Omega)}$ when the bits on the tape are randomly drawn with respect to the uniform distribution.

A set $\mathbf{A}$ is recursive iff both, $\mathbf{A}$ and its complement, are r.e. iff both, $\mathbf{A}$ and its complement, are left-r.e. sets.

2. Reducibilities

$A \leq_{asc} B$ (A is ascendingly reducible to B) iff there is a non-decreasing recursive function f such that $\forall x\, [x \in A \Leftrightarrow f(x) \in B]$.

$A \leq_m B$ (A is many-one reducible to B) iff there is a recursive function f such that $\forall x\, [x \in A \Leftrightarrow f(x) \in B]$.

$A \leq_c B$ (A is conjunctively reducible to B) iff there is a recursive function f such that $\forall x\, [x \in A \Leftrightarrow D_{f(x)} \subseteq B]$.
Canonical index y of D_y: $y = \sum_{z \in D_y} 2^z$.

$A \leq_d B$ (A is disjunctively reducible to B) iff there is a recursive function f such that $\forall x\, [x \in A \Leftrightarrow B \cap D_{f(x)} \neq \emptyset]$.

$A \leq_p B$ (A is positively reducible to B) iff there is a set C with $A \leq_d C \wedge C \leq_c B$.

Examples

If $\mathbf{B} = \{\mathbf{3}, \mathbf{8}\}$ and $\mathbf{A} = \{\mathbf{33}, \mathbf{88}, \mathbf{92}\}$ then $\mathbf{B}$ is many-one reducible to $\mathbf{A}$ via the mapping $\mathbf{x} \mapsto \mathbf{11x}$.

The set $\{\mathbf{3}, \mathbf{8}\}$ is ascendingly reducible to the set $\{\mathbf{33}, \mathbf{88}, \mathbf{92}\}$. However, the set $\{\mathbf{33}, \mathbf{88}, \mathbf{92}\}$ is many-one reducible but not ascendingly reducible to the set $\{\mathbf{3}, \mathbf{8}\}$.

The set $\{\mathbf{x} : \mathbf{2x} \in \mathbf{A} \vee \mathbf{2x} + \mathbf{1} \in \mathbf{A}\}$ is disjunctively reducible to $\mathbf{A}$ but, in general, not conjunctively reducible to $\mathbf{A}$ and also not many-one reducible or ascendingly reducible.

Assume that $\mathbf{A} = \{\mathbf{a_0}, \mathbf{a_1}, \ldots\}$ with $\mathbf{a_0} = \mathbf{1}$ and $\mathbf{a_{n+1}} \in \{\mathbf{2a_n}, \mathbf{2a_n} + \mathbf{1}\}$ for all $\mathbf{n}$. Then every set $\mathbf{B}$ which is positively reducible to $\mathbf{A}$ is also disjunctively reducible to $\mathbf{A}$. Furthermore, if $\mathbf{A}$ is non-recursive then no infinite r.e. set is many-one reducible to $\mathbf{A}$.

3. Closedness

Definition

A set $\mathbf{A}$ is many-one closed left-r.e. iff every $\mathbf{B}$ many-one reducible to $\mathbf{A}$ is a left-r.e. set.
A set $\mathbf{A}$ is ascending closed left-r.e. iff every $\mathbf{B}$ ascendingly reducible to $\mathbf{A}$ is a left-r.e. set.
Similarly for the conjunctive, disjunctive and positive closed left-r.e. sets.

Remarks

If a set is a many-one closed left-r.e. set then it is also an ascending-closed left-r.e. set.
Every r.e. set is a many-one closed left-r.e. set.
Chaitin's Ω is a left-r.e. set but not an ascending closed left-r.e. set.

Example

If $\mathbf{B}$ is r.e. and nonrecursive then the set

$$\mathbf{A} = \{\mathbf{2x} : \mathbf{x} \in \mathbf{B}\} \cup \{\mathbf{2x+1} : \mathbf{x} \notin \mathbf{B}\}$$

is left-r.e. but not an r.e. set. Given an approximation $\mathbf{B_n}$ of $\mathbf{B}$, the approximation $\mathbf{A_n} = \{\mathbf{2x} : \mathbf{x} \in \mathbf{B_n}\} \cup \{\mathbf{2x+1} : \mathbf{x} \notin \mathbf{B_n}\}$ is then a left-r.e. approximation of $\mathbf{A}$.

To see this, assume that $\mathbf{A_{n+1}(y) \neq A_n(y)}$ and $\mathbf{y}$ is the least such number. Then the $\mathbf{x}$ with $\mathbf{2x \leq y \leq 2x+1}$ satisfies that $\mathbf{x \in B_{n+1} - B_n}$. It follows that $\mathbf{2x \in A_{n+1} - A_n}$. So $\mathbf{2x = y}$ and $\mathbf{A_n <_{lex} A_{n+1}}$.

Note that the complement of $\mathbf{B}$ is ascendingly reducible to $\mathbf{A}$. However, the complement of $\mathbf{B}$ is co-r.e. and neither an r.e. nor a left-r.e. set. So $\mathbf{A}$ is neither an ascending closed left-r.e. nor a many-one closed left-r.e. set.

Existence of Closed Set

Theorem

There is a positive closed left-r.e. set which is not an r.e. set.

Basics of Construction of $\mathbf{A}$

- $\mathbf{A}$ is not an r.e. set;
- $\psi_{\mathbf{e}}^{\mathbf{A}}$ is left-r.e. for a numbering $\psi_{\mathbf{0}}, \psi_{\mathbf{1}}, \ldots$ of polynomial time computable positive reducibilities $\psi_{\mathbf{e}}^{\mathbf{A}}$ such that $\mathbf{use}(\psi_{\mathbf{e}}^{\mathbf{A}}, \mathbf{x}) \leq \mathbf{x}$.

Make movable finite sets $\mathbf{X_{e,s}}$ with $\min(\mathbf{X_{e+1,s}}) > \max(\mathbf{X_{e,s}})$ and $\mathbf{X_{e,s}}$ converging to limit $\mathbf{X_e}$ and

$$\mathbf{A} = \tilde{\mathbf{A}} \cup \bigcup_{\mathbf{2e}} \mathbf{X_e} = \mathbb{N} - \bigcup_{\mathbf{2e+1}} \mathbf{X_{2e+1}}$$

for some r.e. set $\tilde{\mathbf{A}}$.

Requirements Used

Goals of the requirements.

- $\mathbf{R_{2d+1}}$: If $\mathbf{W_d}$ intersects infinitely many $\mathbf{X_{2e}}$ then $\mathbf{W_d}$ intersects also some $\mathbf{X_{2e+1}}$;
- $\mathbf{R_{2\langle d,e\rangle}}$: $\psi_{\mathbf{d}}^{\mathbf{A}\cup\mathbf{X_{2e+1}}}(\mathbf{y}) > \psi_{\mathbf{d}}^{\mathbf{A}}(\mathbf{y})$ for some $\mathbf{y} \leq \max(\mathbf{X_{2e+1}})$.

Second requirements are intended to be made true for $\mathbf{e} = \mathbf{d}, \mathbf{d}+\mathbf{1}, \ldots$ so that updates caused by first requirements do preserve that $\psi_{\mathbf{d}}^{\mathbf{A}}$ is a left-r.e. set. The following restraints are utilised.

- $\mathbf{Blk_{2d+1}} = \mathbf{2e+1}$ protect sets $\mathbf{X_{0,s}}, \mathbf{X_{1,s}}, \ldots, \mathbf{X_{2e+1,s}}$ from being changed by any requirement $\mathbf{R_{2d'+1}}$ with $\mathbf{d'} > \mathbf{d}$.
- $\mathbf{Blk_{2d}} = \mathbf{2e+1}$ says that one currently tries to satisfy requirement $\mathbf{R_{2\langle d,e\rangle}}$ and that all requirements of the type $\mathbf{R_{2\langle d,e'\rangle}}$ are satisfied for $\mathbf{e'}$ with $\mathbf{d} \leq \mathbf{e'} < \mathbf{e}$.

A is not an r.e. set

The requirement $\mathbf{R_{2d+1}}$ achieves that $\mathbf{A} \neq \mathbf{W_d}$ by either having that $\mathbf{A} \cap \mathbf{W_d}$ is finite or that $\mathbf{A} \cap \mathbf{W_d}$ intersects some $\mathbf{X_{2e+1}}$.

- $\mathbf{R_{2d+1}}$ requires attention at $\mathbf{s}$ with parameter $\mathbf{e}$ if $\mathbf{W_{d,s}}$ intersects $\mathbf{X_{2e+2,s}}$ but not any $\mathbf{W_{2e'+1,s}}$ with $\mathbf{e'} \in \{\mathbf{0}, \mathbf{1}, \ldots, \mathbf{e}\}$ and $\mathbf{Blk_{2d'+1}} < \mathbf{2e+1}$ for all $\mathbf{d'} < \mathbf{d}$ and $\mathbf{Blk_{2d'}} \neq \mathbf{2e+1}$ for all $\mathbf{d'} < \mathbf{d}$.
- $\mathbf{R_{2d+1}}$ acts by letting $\mathbf{X_{c,s+1}} = \mathbf{X_{c,s}}$ for $\mathbf{c} \leq \mathbf{2e}$, $\mathbf{X_{c,s+1}} = \mathbf{X_{c+1,s}}$ for $\mathbf{c} \geq \mathbf{2e+1}$, $\tilde{\mathbf{A}}_{\mathbf{s+1}} = \tilde{\mathbf{A}}_{\mathbf{s}} \cup \mathbf{X_{2e+1,s}}$.
- $\mathbf{Blk_{2d'+1}}$ is set to $\mathbf{0}$ for all $\mathbf{d'} > \mathbf{d}$. All $\mathbf{Blk_{2d'}} \geq \mathbf{2e+1}$ are reset to $\max\{\mathbf{2e+1}, \mathbf{2d'+1}\}$.

A is a positively closed left-r.e. set

The requirement $\mathbf{R_{2\langle d,e\rangle}}$ achieves that $\psi_{\mathbf{d}}^{\mathbf{A}}$ is a left-r.e. set.

- $\mathbf{R_{2\langle d,e\rangle}}$ requires attention at $\mathbf{s}$ if $\mathbf{Blk_{2d} = 2e + 1}$ and $\max(\mathbf{X_{2e+1,s}}) \leq \mathbf{s}$ and there is $\mathbf{y \leq s}$ with $\psi^{\mathbf{A_s} \cup \{\mathbf{x}:\min(\mathbf{X_{2e+1,s}}) \leq \mathbf{x} \leq \mathbf{s}\}}(\mathbf{y}) > \psi^{\mathbf{A_s}}(\mathbf{y})$.
- If $\mathbf{R_{2\langle d,e\rangle}}$ acts then $\mathbf{X_{2e+1,s+1}}$ is the union of $\mathbf{X_{2c+1,s}}$ for $\mathbf{c = e, e+1, \ldots, s}$ and $\mathbf{\tilde{A}_{s+1}}$ it the union of $\mathbf{\tilde{A}_s}$ and $\mathbf{X_{2c+2,s}}$ for $\mathbf{c = e, e+1, \ldots, s-1}$ and $\mathbf{X_{c,s+1} = X_{c,s}}$ for $\mathbf{c \leq 2e}$ and $\mathbf{X_{c,s+1} = x_{c+(2s-2e),s}}$ for $\mathbf{c \geq 2e+2}$.
- $\mathbf{Blk_{2d}}$ is updated to $\mathbf{2e+3}$ and if $\mathbf{d' > d \wedge Blk_{2d'} > 2e+1}$ then $\mathbf{Blk_{2d'+1}}$ is updated to $\max\{\mathbf{2e+3, 2d'+1}\}$. For all $\mathbf{d'}$ with $\mathbf{Blk_{2d'+1} = 2e'+1}$, $\mathbf{Blk_{2d'+1}}$ is updated to the value $\mathbf{2e''+1}$ with $\mathbf{X_{2e'+1,s} \subseteq X_{2e''+1,s+1}}$.

4. Cohesive and r-cohesive sets

Definition [Friedberg 1958, Myhill 1956]
A set $\mathbf{A}$ is cohesive iff $\mathbf{A}$ is infinite and for every r.e. set $\mathbf{B}$, either $\mathbf{A} \cap \mathbf{B}$ or $\mathbf{A} - \mathbf{B}$ is finite.
A set $\mathbf{A}$ is r-cohesive iff $\mathbf{A}$ is infinite and for every recursive set $\mathbf{B}$, either $\mathbf{A} \cap \mathbf{B}$ or $\mathbf{A} - \mathbf{B}$ is finite.

Every cohesive set is r-cohesive but not vice versa.
No r.e. set is r-cohesive: If $\mathbf{A}$ is r.e. and infinite then $\mathbf{A}$ has an infinite recursive subset $\mathbf{B}$ which in addition satisfies that $\mathbf{A} - \mathbf{B}$ is infinite.

Theorem [Soare 1969]
There is a cohesive left-r.e. set.

Theorem
Every r-cohesive left-r.e. set is actually an ascending closed left-r.e. set.

R-Cohesive Left-R.E. is Asc Closed

Let $\mathbf{A}$ be an r-cohesive and left-r.e. and let $\mathbf{B}$ be ascendingly reducible to $\mathbf{A}$ via $\mathbf{f}$. Assume that $\mathbf{B}$ is infinite; otherwise $\mathbf{B}$ is trivially a left-r.e. set. Then the range of $\mathbf{f}$ is infinite and recursive. Hence almost all elements of $\mathbf{A}$ are in the range of $\mathbf{f}$, say all above $\mathbf{u}$. Given a left-r.e. enumeration $\mathbf{A_n}$, there is a recursive sequence $\mathbf{n_0, n_1, \ldots}$ such that $\mathbf{A_{n_0}(x) = A(x)}$ for all $\mathbf{x \leq u}$ and $\mathbf{A_{n_m}(x) = 1 \Rightarrow x \in range(f)}$ for all $\mathbf{x \in \{u, u+1, \ldots, u+m\}}$. Now define

$$\mathbf{B_m(x)} = \begin{cases} A_{n_m}(f(x)) & \text{if } f(x) \leq u+m; \\ 0 & \text{otherwise.} \end{cases}$$

Then the sequence of the $\mathbf{B_m}$ witnesses that $\mathbf{B}$ is a left-r.e. set: Given $\mathbf{y \in B_m - B_{m+1}}$, then $\mathbf{f(y) \in A_{n_m} - A_{n_{m+1}}}$ and there is an $\mathbf{z < f(y)}$ with $\mathbf{z \in A_{n_{m+1}} - A_{n_m}}$. Now $\mathbf{z > u}$ and $\mathbf{z = f(x)}$ for some $\mathbf{x}$. Now $\mathbf{x < y}$ and $\mathbf{x \in B_{m+1} - B_m}$.

Closedness of Cohesive Sets

Theorem
There is a many-one closed left-r.e. set which is cohesive.

Theorem
There is a left-r.e. cohesive set which is not a many-one closed left-r.e. set.

Both sets are constructed using a variant of Friedberg's construction of a maximal set using e-states.

Supersets of Maximal Sets

A set $\mathbf{A}$ is maximal / r-maximal iff it is r.e. and its complement is cohesive / r-cohesive, respectively.

In the following, let $\mathbf{a_0, a_1, \ldots}$ be the elements of the complement of $\mathbf{A}$ in ascending order.

Theorem

If $\mathbf{A}$ is maximal and $\mathbf{B}$ is left-r.e. then $\mathbf{A \cup \{a_n : n \in B\}}$ is a many-one closed left-r.e. set.
If $\mathbf{A}$ is only r-maximal then the $\mathbf{B}$ constructed this way is still ascending closed left-r.e. but might fail to be a many-one closed left-r.e. set.

5. Weakly 1-generic sets

Definition

A set $\mathbf{A}$ is called 1-generic iff for every partial-recursive extension function $\mathbf{F}$ either there are $\mathbf{n}, \mathbf{m}$ with $\mathbf{F}(\mathbf{A}(0)\mathbf{A}(1)\ldots\mathbf{A}(\mathbf{n})) = \mathbf{A}(0)\mathbf{A}(1)\ldots\mathbf{A}(\mathbf{m})$ or there is an $\mathbf{n}$ such that $\mathbf{F}(\sigma)$ is undefined for all σ extending $\mathbf{A}(0)\mathbf{A}(1)\ldots\mathbf{A}(\mathbf{n})$.

A set $\mathbf{A}$ is called weakly 1-generic iff for every recursive extension function $\mathbf{F}$ there are $\mathbf{n}, \mathbf{m}$ with $\mathbf{F}(\mathbf{A}(0)\mathbf{A}(1)\ldots\mathbf{A}(\mathbf{n})) = \mathbf{A}(0)\mathbf{A}(1)\ldots\mathbf{A}(\mathbf{m})$.

It is known that left-r.e. sets can only be weakly 1-generic but not 1-generic.

Theorem

There is a many-one closed left-r.e. weakly 1-generic set.

6. Kolmogorov complexity

The Kolmogorov complexity $\mathbf{C}(\sigma)$ of a string σ with respect to a universal machine $\mathbf{U}$ is the length of the shortest $\mathbf{p}$ such that $\mathbf{U}(\mathbf{p}) = \sigma$. If one changes the universal machine then there is the constant $\mathbf{c}$ such that the complexity of each string does not change more than $\mathbf{c}$.

Many investigations investigate the initial segment complexity $\mathbf{n} \mapsto \mathbf{C}(\mathbf{A}(\mathbf{0})\mathbf{A}(\mathbf{1})\ldots\mathbf{A}(\mathbf{n}))$ for sets $\mathbf{A}$.

Theorem
The initial segment complexity of any ascending closed left-r.e. set $\mathbf{A}$ is sublinear.

Proof of Sublinearity

Given $\mathbf{c}$, $\mathbf{A}$ can be viewed as the join of $\mathbf{c}$ sets $\mathbf{B_1, B_2, \ldots, B_c}$ and each set $\mathbf{B_d}$ is ascending reducible to $\mathbf{A}$. Hence every $\mathbf{B_d}$ is left-r.e. and one now considers $\mathbf{A(0)A(1)\ldots A(c \cdot n - 1)}$.

There is a $\mathbf{d}$ on which the left-r.e. approximation $\mathbf{B_{d,s}}$ below $\mathbf{n}$ converges last at time $\mathbf{t}$; this $\mathbf{t}$ can be computed from $\mathbf{B_d(0)B_d(1)\ldots B_d(n-1)}$. Now $\mathbf{A(c \cdot m + b) = B_b(m)}$ for all $\mathbf{b \in \{1, \ldots, c\}}$ and all $\mathbf{m < n}$, hence $\mathbf{A(c \cdot m + b) = B_{b,t}(m)}$ for all $\mathbf{t}$ computed from $\mathbf{B_d(0)B_d(1)\ldots B_d(n-1)}$. Thus the first $\mathbf{c \cdot n}$ bits of $\mathbf{A}$ can be computed from $\mathbf{d}$ and the first $\mathbf{n}$ bits of $\mathbf{B_d}$.

So, for every $\mathbf{c}$ there is a constant $\mathbf{k}$ with $\mathbf{C(A(0)A(1)\ldots A(n-1)) \leq n/c + k}$.

More on Kolmogorov complexity

Theorem
Given a recursive increasing and unbounded function $\mathbf{f}$ there is a ascending closed left-r.e. set $\mathbf{A}$ with $\mathbf{C(A(0)A(1)\ldots A(n)) \geq n/f(n)}$ for almost all $\mathbf{n}$.

Theorem
Given a recursive increasing and unbounded function $\mathbf{f}$ there is a many-one closed left-r.e. set $\mathbf{A}$ with $\mathbf{C(A(0)A(1)\ldots A(n)) \geq n/f(n)}$ for infinitely many $\mathbf{n}$.

Theorem
If a set $\mathbf{A}$ is conjunctively closed left-r.e. or disjunctively closed left-r.e. then for every $\varepsilon > \mathbf{0}$ and almost all $\mathbf{n}$, $\mathbf{C(A(0)A(1)\ldots A(n)) \leq (2+\varepsilon)\cdot \log(n)}$.

Remark
These theorems prove also that there is a many-one closed set which is not conjunctively closed or disjunctively closed.

Summary

This talk introduced r closed left-r.e. for reducibilities r.

For sets, the implications r.e. $\Rightarrow$ positive closed left-r.e. $\Rightarrow$ many-one closed left-r.e. $\Rightarrow$ ascending closed left-r.e. $\Rightarrow$ left-r.e. hold and no arrow can be reversed.

Cohesive, r-cohesive and weakly 1-generic sets are not r.e. but can be many-one closed left-r.e.; furthermore there is an ascending closed left-r.e. set which is cohesive and not many-one closed left-r.e.; however every cohesive left-r.e. set is also an ascending closed left-r.e. set.

The initial segment complexity of ascending closed left-r.e. sets is sublinear and can be kept near to linear; for many-one closed left-r.e. sets, one can obtain such bounds infinitely often; for conjunctive / disjunctive / positive closed left-r.e. sets the bound is logarithmic.

Open Problems

Problem 1.
Are conjunctively closed left-r.e. sets also disjunctively closed left-r.e. and vice versa?

Problem 2.
The many-one closed left-r.e. sets are closed under join and many-one reducibility. Is there a greatest such degree? That is, is there is a set $\tilde{\mathbf{K}}$ such that $\mathbf{A}$ is many-one closed left-r.e. iff $\mathbf{A} \leq_{\mathbf{m}} \tilde{\mathbf{K}}$?

Problem 3.
Is every enumeration closed left-r.e. set already an r.e. set?