

Generalization of van Lambalgen's theorem and blind randomness for conditional probability

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Lambalgen's theorem

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Conditional probability

Computability of conditional probability

Vovk and V'yugin 93

Takahashi 2008, 2011

Blind randomness

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Notation: $\Omega := \{0, 1\}^\infty \ni x^\infty, y^\infty$, and $S := \{0, 1\}^* \ni x, y$.

P_U : unifrom probability.

Theorem 1 (Lambalgen)

(x^∞, y^∞) is ML-random w.r.t. $P_U \times P_U$

$\Leftrightarrow y^\infty$ is ML-random w.r.t. P_U

x^∞ is ML-random w.r.t. P_U relative to y^∞ .



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P : computable probability on $X \times Y, X = Y = \Omega$.

$P(x, y) := P(\Delta(x) \times \Delta(y)), \Delta(x) := \{x\omega \mid \omega \in \Omega\}$.

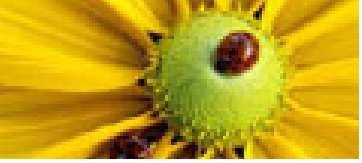
$P_X(x) := P(x, \Omega), P_Y(y) := P(\Omega, y)$: marginal distribution

Theorem 2 (Takahashi 2008)

$P(x \mid y^\infty) := \lim_{y \rightarrow y^\infty} P(x \mid y)$ exists for all $x \in S$ and for all

ML-random y^∞ w.r.t. P_Y .

$P(\cdot \mid y^\infty)$ is defined for all *ML-random* y^∞ w.r.t. P_Y .



Computability of conditional probability

Computability of conditional probability

■ uniform computability

$$\exists A \forall y^\infty \in \mathcal{R}^{P_Y} \forall x, k \exists y \sqsubset y^\infty |A(x, k, y) - P(x | y^\infty)| < \frac{1}{k}.$$

■ computability for fixed y^∞ .

$$\exists A \forall x, k \exists y \sqsubset y^\infty |A(x, k, y) - P(x | y^\infty)| < \frac{1}{k}.$$

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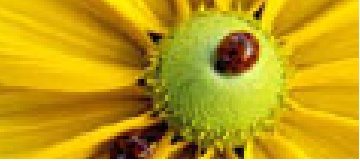
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Theorem 3 (Vovk and V'yugin) *Let P be a computable probability on $X \times Y$, $X = Y = \Omega$. Under the assumptions that*

- (i) conditional probabilities exist for all parameters and*
- (ii) they are uniformly computable for all parameters,*

$(x^\infty, y^\infty) \in \Omega^2$ is ML-random w.r.t. P on $X \times Y$ iff y^∞ is ML-random w.r.t. P_Y and x^∞ is ML-random w.r.t. $P(\cdot|y^\infty)$ relative to y^∞ .

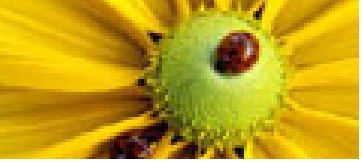
Theorem 4 (Roy 2011) *There is a computable probability whose conditional probability is not uniformly computable.*



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Theorem 5 (Takahashi 08, 11) *Let P be a computable probability on $X \times Y$, $X = Y = \Omega$. Fix a ML-random y^∞ w.r.t. P_Y . If the conditional probability $P(\cdot|y^\infty)$ is computable relative to y^∞ then,*
 $(x^\infty, y^\infty) \in \Omega^2$ is ML-random w.r.t. P on $X \times Y$ iff y^∞ is ML-random w.r.t. P_Y and x^∞ is ML-random w.r.t. $P(\cdot|y^\infty)$ relative to y^∞ .



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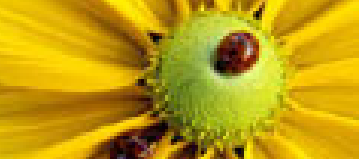
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P : probability on Ω .

- blind test w.r.t. P : U r.e., $U_n := \{x \mid (x, n) \in U\}$,
 $\forall n U_n \supseteq U_{n+1}$, $P(\tilde{U}_n) < 2^{-n}$.
- x^∞ is blind random w.r.t. P iff $x^\infty \notin \bigcap_n \tilde{U}_n$ for all blind test U w.r.t. P [Hanssen 10, Bienvenu et.al 11].
If probability is not computable, the existence of universal test is not assured.
- blind test w.r.t. $P(\cdot|y^\infty)$: U r.e. set relative to y^∞ ,
 $U_n := \{x \mid (x, n) \in U\}$,
 $\forall n U_n \supseteq U_{n+1}$, $P(\tilde{U}_n \mid y^\infty) < 2^{-n}$.
- x^∞ is blind random w.r.t. $P(\cdot|y^\infty)$ iff $x^\infty \notin \bigcap_n \tilde{U}_n$ for all blind test U w.r.t. $P(\cdot|y^\infty)$

Note: Computability assumptions on P and $P(\cdot|y^\infty)$ are not necessary.



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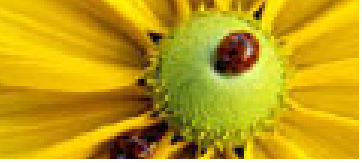
Theorem 6 (main theorem) *Let P be a computable probability on $X \times Y$, $X = Y = \Omega$. Under Assumption 1 (see below), we have $(x^\infty, y^\infty) \in \Omega^2$ is ML-random w.r.t. P on $X \times Y$ iff y^∞ is ML-random w.r.t. P_Y and x^∞ is blind-random w.r.t. $P(\cdot|y^\infty)$ relative to y^∞ .*



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- The if part of the proof follows from the proof of Theorem 4.2 in [Takahashi 08] since computability of the conditional probability is not assumed in the proof.
- The proof of the only if part is similar to that of Theorem 3.3 in [Takahashi 11], however since we do not assume the computability of conditional probability, we need modify the proof.



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1. Fix a ML-random \bar{y}^∞ w.r.t. P_Y .
2. Let $V \subseteq S$ be r.e. relative to \bar{y}^∞ and $P(\tilde{V}|\bar{y}^\infty) < \epsilon$, where ϵ is a rational number.
3. From V , we construct r.e. $U \subseteq S^2$ as follows:

$$U \text{ is r.e., } \tilde{U}_{\bar{y}^\infty} = V, \text{ and } P(\tilde{U}) < 2\epsilon. \quad (1)$$

4. B: partial comp.

$$V = \{x \mid \exists i, y \sqsubset y^\infty \ B(i, y) = x\}$$

$$W = \{(x, y) \mid \exists i, y \sqsubset y^\infty \ B(i, y) = x\}$$

Then W is r.e. and $\tilde{W}_{\bar{y}^\infty} = V$.



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For $A \subset S^2$, $\tilde{A} = \cup_{(x,y) \in A} \Delta(x, y)$.

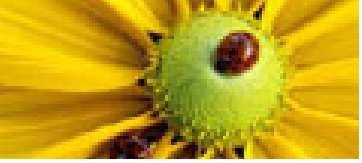
$W' := \{(x^1, y^1), (x^2, y^2), \dots\}$ non-overlapping r.e. s.t. $\tilde{W} = \tilde{W}'$

$W'_n = \{(x^1, y^1), \dots, (x^n, y^n)\}$

$Y_n = \{y \mid \Delta(y) \subseteq \cap_i A_i, A_i \in \{\Delta(y^i), \Delta(y^i)^c\}, 1 \leq i \leq n\}$

$$U_n := \{(x, y) \mid \sum_{x:(x,z) \in W'_n, z \sqsubseteq y} P(x|y) < \epsilon, y \in Y_n\}$$

$$U := \cup_n U_n$$



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$$\widetilde{U}_n U = \cup_n \tilde{U}_n = \cup_n (\tilde{U}_n \setminus \tilde{U}_{n+1}) \cup \liminf_n \tilde{U}_n,$$
$$(\tilde{U}_n \setminus \tilde{U}_{n+1}) \cap (\tilde{U}_m \setminus \tilde{U}_{m+1}) = \emptyset, \quad n \neq m \text{ and,}$$
$$\cup_n (\tilde{U}_n \setminus \tilde{U}_{n+1}) \cap \liminf_n \tilde{U}_n = \emptyset,$$

Then

- U is r.e.,
- $P(\liminf_n \tilde{U}_n) \leq \epsilon,$
- $V = (\liminf_n \tilde{U}_n)_{y^\infty}.$



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$$P(\tilde{U}) = \sum_n P(\tilde{U}_n \setminus \tilde{U}_{n+1}) + \liminf_n \tilde{U}_n < \sum_n P(\tilde{U}_n \setminus \tilde{U}_{n+1}) + \epsilon.$$

Let $f : \mathbb{Q} \rightarrow \mathbb{N}$ s.t. $\sum_{n > f(\epsilon)} P(\tilde{U}_n \setminus \tilde{U}_{n+1}) < \epsilon.$

Then $P(\tilde{U}) < 2\epsilon.$

Assumption 1: f is computable

Under the assumption $\cup_{n > f(\epsilon)} \tilde{U}_n$ satisfies (1).



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- [1] Bjørn Kjos Hanssen. The probability distribution as a computational resource for randomness testing. *Journal of Logic and Analysis*, 2(10):1-13, 2010.
- [2] L. Bienvenu, P. Gacs, M. Hoyrup, C. Rojas, and A. Shen. Algorithmic tests and randomness with respect to a class of measures, 2011. arxiv:1103.1529v2.
- [3] D. M. Roy. *Computability, inference and modeling in probabilistic programming*. PhD thesis, MIT, 2011.
- [4] H. Takahashi. On a definition of random sequences with respect to conditional probability. *Inform. and Compt.*, 206:1375–1382, 2008.
- [5] H. Takahashi. Algorithmic randomness and monotone complexity on product space. *Inform. and Compt.*, 209:183–197, 2011.
- [6] M. van Lambalgen. *Random sequences*. PhD thesis, Universiteit van Amsterdam, 1987.
- [7] V. G. Vovk and V. V. V'yugin. On the empirical validity of the Bayesian method. *J. R. Stat. Soc. B*, 55(1):253–266, 1993.