Things that can be made into themselves

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The birth of mirth

Previous work has shown that algorithms can be used to characterize random sequences.

Question

Can we use computers to identify humorous expressions?

While there is no absolute notion of "humor," clearly some formulas are more humorous than others.

Example

Let $\epsilon < 0$.

We wish to capture the notion of humor in formal languages so that computers can process funny things from the Internet as well as generate comedies.

Computational semantics

Some forms of humor require and exploit ambiguity.

Example

Is it funny to add the phrase, "That's what she said" to the end of a given sentence? Kiddon and Brun show that computers can do better than just search for sentence words which women are likely to use.

Kao, Levy, and Goodman investigated computer identification of puns, e.g.

> *"The magician got so mad he pulled his hare out."*

Can we substitute ambiguity with self-reference?

> *"The magician got so mad he pulled himself out."*

We would like to mix syntax and semantics by applying self-reference.

A strange phenomenon

Dougherty's Paradox

- There are infinitely many infinite sets, but
- there are not finitely many finite sets.

The infinite sets "can be made into themselves," but the finite sets cannot. We explore this self-referential mixing of syntax and semantics in the context of recursion theory.

Definition

We write $A \leq_{\text{lex}} B$ if either $A = B$ or the least element x of the symmetric difference satisfies $x \in B$.

We do not identify the characteristic sequence $10000000\ldots$ with $01111111\ldots$.

Enumerations of left-r.e. sets (not reals)

Definition

A set A is *left-r.e.* iff there is a uniformly recursive approximation $A_0, A_1, \ldots$ of A such that $A_s \leq_{\mathrm{lex}} A_{s+1}$ for all s.

Definition

A *left-r.e. numbering* α is a recursive function from natural numbers to left-r.e. sets given by

$$e \mapsto \lim_{s \to \infty} \alpha_{e,s} = \alpha_e$$

where

- $\alpha_{e,s}$ is a recursive set, uniform in e and s, and
- $\alpha_{e,s} \leq_{\mathrm{lex}} \alpha_{e,s+1}$ for all s.

Universal left-r.e. numberings

Definition

A left-r.e. numbering is called *universal* if its range includes all left-r.e. sets.

Suppose

$$\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \ldots$$

is a universal left-r.e. numbering. Some of the α_i's will be Martin-Löf random, say $\alpha_1, \alpha_7, \alpha_{15}, \ldots$.

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Is $\{1, 7, 15, \ldots\}$ a Martin-Löf random set?

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Making things into themselves

Definition

A class of sets $\mathcal{C}$ *can be made into itself* if there exists a universal left-r.e. numbering α such that

$$\{e : \alpha_e \in \mathcal{C}\} \in \mathcal{C}.$$

For example, consider singleton classes. A singleton left-r.e. class $\{A\}$ *can be made into itself* if there exists a universal left-r.e. numbering β such that

$$\{e : \beta_e = A\} = A.$$

Randoms can be random

Theorem (Stephan, Teutsch)

The Martin-Löf random sets can be made into themselves.

The same is true for the classes of:

- computably randoms,
- Schnorr randoms,
- Kurtz randoms,
- bi-immune sets
- immune sets,
- nonrecursive sets.

On the other hand,

Theorem (Stephan, Teutsch)

The left-r.e. Martin-Löf random sets cannot be made into themselves.

The random set witnessing that the Martin-Löf randoms get made into themselves cannot be Chaitin's Ω, but it can be K-recursive.

genericity

A set of binary strings A is called *dense* if for every string σ there exists $\tau \in A$ extending σ.

Definition

A set is *weakly 1-generic* if it has a prefix in every dense r.e. sets of binary strings.

Theorem (Stephan, Teutsch)

The weakly 1-generic sets can be made into themselves.

Theorem (Stephan, Teutsch)

No left-r.e. numbering makes both the weakly 1-generic sets into themselves and the Martin-Löf random sets into themselves.

Things which cannot be made into themselves

The finite sets is not the only class which cannot be made into itself.

Lemma (Stephan, Teutsch)

The following classes have no left-r.e. numbering:

- *the non-r.e. left-r.e. sets,*
- *the non-recursive left-r.e. sets.*

Although somewhat disappointing, we get the following consequence.

Corollary (Stephan, Teutsch)

The following classes cannot be made into themselves:

- *the r.e. sets,*
- *the co-r.e. sets, and*
- *the recursive sets.*

garden variety numberings

Definition

A left-r.e. numbering α is called *acceptable* if for every left-r.e. numbering β there exists a recursive function f such that $\alpha_{f(e)} = \beta_e$ for all e.

A set A is called *autoreducible* if for all x, whether x is a member of A can be effectively determined by querying A at positions other than x.

Proposition (Stephan, Teutsch)

Any acceptable left-r.e. numbering makes the following classes into themselves:

- *the autoreducible sets,*
- *the non-random sets,*

but not these:

- *the non-autoreducible sets,*
- *the random sets.*

Singleton classes

Theorem (Stephan, Teutsch)

Let α be an acceptable universal left-r.e. numbering. Then for every set B, $\{e : \alpha_e = B\} \neq B$.

The singleton classes which can be made into themselves admit a nice characterization.

Theorem (Stephan, Teutsch)

A singleton left-r.e. class $\{A\}$ can be made into itself iff $A \neq \emptyset$ and there exists an infinite, r.e. set B such that $A \cap B = \emptyset$.

Indifferent sets

Definition (Figueira, Miller, Nies)

An infinite set I is called *indifferent for a set A with respect to $\mathcal{C}$* if for any set X,

$$X \triangle A \subseteq I \implies X \in \mathcal{C}.$$

Theorem (Figueira, Miller, Nies)

Every low Martin-Löf random set has an infinite co-r.e. set which is indifferent with respect to the class of Martin-Löf randoms.

Can the same hold for left-r.e. random sets?

Question (Figueira, Miller, Nies)

Does Chaitin's Ω have an infinite co-r.e. set which is indifferent for the class of Martin-Löf randoms?

A partial solution

In order to make things into themselves, we add "retraceable" to the result of Figueira, Miller, and Nies.

Definition

A set $A = \{a_0 < a_1 < a_2 < \cdots\}$ is *retraceable* if there exists a partial-recursive function f satisfying $f(a_{n+1}) = a_n$ for all n.

The result then becomes:

Theorem (Stephan, Teutsch)

Every low Martin-Löf random set has a co-r.e. indifferent set which is retraceable by a recursive function.

It is impossible to replace "low" with "left-r.e." in the above theorem.

Post's Problem

In the beginning, recursion theorists only knew of enumerable objects which were similar to the halting set, K.

Question (Post, 1944)

Does there exist a nonrecursive r.e. set A with Turing degree below K?

- A set is called *immune* if it is infinite but contains no r.e. subset. An r.e. set is *simple* if its complement is immune.
- Post introduced simple sets as having "thin" complement with many-one degree strictly below K.
- By making even thinner complements, he proposed to get an r.e. set with *Turing* degree strictly below K.

Almost everywhere equal

Notation

We write

- $A \subseteq^* B$ if almost all elements of A are members of B, and
- $A \subset^* B$ if $A \subseteq^* B$ and also $B - A$ is infinite.

Myhill first investigated the structure of r.e. sets under $\subset^*$.

Definition

An r.e. set M is called *minimal r.e.* if

- $\emptyset \subset^* M$, and
- there is no r.e. set E with $\emptyset \subset^* E \subset^* M$.

Unforutnately, minimal r.e. sets do not exist: one can always enumerate every other element of a given r.e. set.

Post's program fails

Definition (Myhill, 1956)

An r.e. set M is *maximal r.e.* if

- $M \subset^* \mathbb{N}$, and
- there is no r.e. set E with $M \subset^* E \subset^* \mathbb{N}$.

Theorem (Friedberg, 1958)

There exists a maximal r.e. set.

Theorem (Lachlan 1968, Yates 1965)

A maximal set can be Turing complete.

This destroys Post's Program: maximal sets have the thinnest possible complement among infinite r.e. sets.

Operations on r.e. and left r.e. sets

Similarities and differences.

	r.e. sets	left-r.e. sets
closed under join	✓	✓
closed under complement	×	×
closed under intersection	✓	×
closed under union	✓	×

Proposition

Either Chaitin's Ω intersected with either the even numbers or Ω intersected with the odd number is not left-r.e.

Proof.

Otherwise the initial segment complexity of Ω would have sublinear order. Observe which half converges faster on each prefix of Ω. □

Maximal and minimal left-r.e. sets

Definition (Stephan, Teutsch)

A left-r.e. set M is *minimal left-r.e.* if

- $\emptyset \subset^* M$, and
- there is no left-r.e. set E with $\emptyset \subset^* E \subset^* M$.

A left-r.e. set M is *maximal left-r.e.* if

- $M \subset^* \mathbb{N}$, and
- there is no left-r.e. set E with $M \subset^* E \subset^* \mathbb{N}$.

Theorem (Stephan, Teutsch)

Maximal left-r.e. sets exist. Minimal left-r.e. exist, too.

Theorem (Stephan, Teutsch)

An r.e. set cannot be maximal left-r.e.

Cohesive sets

Definition

An infinite set C is *cohesive* if for every r.e. set X either $C \cap X$ or $C \cap \overline{X}$ is finite.

Proposition

An set is r.e. maximal iff its complement is cohesive.

By the theorem of Friedberg, cohesive sets exist. An analogous class for left-r.e. sets fails to exist.

Theorem (Stephan, Teutsch)

Let A be any infinite left-r.e. set. Then there exists an infinite left-r.e. set E such that $A \cap E$ and $A \cap \overline{E}$ are both infinite.

Maximal and minimal singletons

Theorem (Stephan, Teutsch)

There is a minimal left-r.e. set A such that $\{A\}$ can be made into itself, but there is no maximal left-r.e. set B such that $\{B\}$ can be made into itself.

Proof (maximal case).

By the theorem on singleton classes, it suffices to show that there is no infinite recursive set R disjoint from B. We may assume $B \cup R$ is coinfinite because otherwise B is the complement of a recursive set and not maximal. Let $B_0, B_1, \ldots$ be a left-r.e. approximation of B. Now one can select a sequence of stages $s_0, s_1, \ldots$ such that $B_{s_t} \cap \{0, 1, \ldots, t\}$ is disjoint from R. Hence $E_t = (B_{s_t} \cap \{0, 1, \ldots, t\}) \cup R$ is a recursive left-r.e. approximation of $B \cup R$ which witnesses that B is not a maximal left-r.e. set. □

Maximal and minimal classes

Theorem (Stephan, Teutsch)

Neither the class of minimal left-r.e. sets nor the class of maximal left-r.e. sets can be made into itself.

Proof (maximal case).

Let B be a maximal let-r.e. set and let Q be the index set of the maximal left-r.e. sets in a given numbering α. Now each join $B \oplus W_e$ is left-r.e. and is a maximal left-r.e. set iff W_e is cofinite. There exists is a K'-recursive mapping which finds for each e an index d with $B \oplus W_e = \alpha_d$; hence one can, relative to K', many-one reduce the index set of the cofinite sets to Q. As the index set of the cofinite sets is not K'-recursive, Q also cannot be K'-recursive; hence Q cannot be left-r.e. and in particular is not a maximal left-r.e. set. □

THE END. Thanks!

The r.e. *inclusion problem* is

$$\mathrm{INC}_\varphi = \langle i, j \rangle : W_i^\varphi \subseteq W_j^\varphi \}.$$

Kummer showed that there is a numbering which makes the inclusion problem recursive in the halting set and asked whether the inclusion problem can be r.e. We get a negative answer for the left-r.e. case.

Theorem

For every universal left-r.e. numbering α,

1. INC_α *is not r.e. and*
2. $\mathrm{INC}_\alpha \geq_{\mathrm{T}} K$.

Question

Does there exist a numbering α for the left-r.e. sets such that $\mathrm{INC}_\alpha \equiv_{\mathrm{T}} K$? In particular, can we make INC_α to be left-r.e.?

Complexity of the lexicography problem

Consider the related relation

$$\mathrm{LEX}_\alpha = \{\langle i, j\rangle : \alpha_i \leq_{\mathrm{lex}} \alpha_j\}.$$

Any Friedberg numbering α makes LEX_α recursive in the halting set. Moreover,

Theorem

There exists a universal left-r.e. numbering α such that LEX_α is an r.e. relation.