On Stability Property of Probability Laws with Respect to Small Violations of Algorithmic Randomness

Vladimir V. V'yugin

Institute for Information Transmission Problems
Russian Academy of Sciences

logo

Martin-Löf random sequences

Martin-Löf test of randomness is a r.e. sequence $\{U_n\}$ of effectively open sets such that $P(U_n) \leq 2^{-n}$ for all n

ω passes test $\{U_n\}$ if $\omega \notin U_n$ for almost all n

ω is Martin-Löf random (w.r.to uniform L) if it passes all Martin-Löf tests

$K(x) = \min\{|p| : x \subseteq F(p))\}$ - monotonic (or prefix) complexity

$K(\omega^n) \geq n - O(1) \iff \omega$ is Martin-Löf random

We use notation: $\omega^n = \omega_1 \ldots \omega_n$

logo

Probability laws

Pointwise form of probability law:

$$K(\omega^n) \geq n - O(1) \Longrightarrow A(\omega).$$

Law of large numbers for symmetric Bernoulli scheme:

$$K(\omega^n) \geq n - O(1) \Longrightarrow \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \omega_i = 1/2.$$

Law of iterated logarithm:

$$K(\omega^n) \geq n - O(1) \Longrightarrow \limsup_{n\to\infty} \frac{\sum_{i=1}^{n} \omega_i - n/2}{\sqrt{\frac{1}{2} n \ln\ln n}} = 1.$$

logo

Algorithmic version of the Birkhoff's ergodic theorem

- A transformation $T : \Omega \to \Omega$ preserves a measure P if $P(T^{-1}(A)) = P(A)$ for all A.
- A measurable subset $A \subseteq \Omega$ is invariant with respect to T if $T^{-1}(A) = A$ modulo a set of measure 0.
- T is ergodic if $P(A) = 0$ or $P(A) = 1$ for each invariant A.

Theorem

For any computable transformation T preserving the uniform measure and computable bounded observable f

$$K(\omega^n) \geq n - O(1) \Rightarrow \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i\omega) = \hat{f}(\omega)$$

for some $\hat{f}$ ($= E(f)$ for ergodic T).

Local stability of probability laws

Law of large numbers (Schnorr (1973):

$$K(\omega^n) \geq n - \alpha(n) - O(1) \Longrightarrow \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \omega_i = 1/2,$$

where $\alpha(n) = o(n)$ as $n \to \infty$.

Law of iterated logarithm (Vovk (1986)):

$$K(\omega^n) \geq n - \alpha(n) - O(1) \Longrightarrow \limsup_{n\to\infty} \frac{\sum_{i=1}^{n} \omega_i - n/2}{\sqrt{\frac{1}{2} n \ln\ln n}} = 1,$$

where $\alpha(n) = o(\ln\ln n)$ as $n \to \infty$.

Sufficient condition for stability of a probability law

Why such stability?

Schnorr test of randomness is a Martin-Löf test of randomness U_n such that the measure $L(U_n)$ is the computable function of n.

Theorem

For any Schnorr test of randomness $\mathcal{T}$ a computable unbounded function $\rho(n)$ exists such that for any infinite sequence ω if $K(\omega^n) \geq n - \rho(n) - O(1)$ then the sequence ω passes the test $\mathcal{T}$.

Effective convergence almost surely (a.s.)

$f_n(\omega)$ – computable sequence of functions of type: $\Omega \to [a, b]$.
$f_n(\omega) \to f(\omega)$ a.s. effectively converges if a computable function $N(\delta, \varepsilon)$ exists such that $L\{\omega : \sup_{n \geq N(\delta,\varepsilon)} |f_n(\omega) - f(\omega)| > \delta\} < \varepsilon$ for all positive rational numbers δ and ε.

Theorem

If $f_n(\omega) \to f(\omega)$ a.s. effectively converges then a Schnorr test of randomness exists such that if a sequence ω passes this test then $\lim_{n\to\infty} f_n(\omega) = f(\omega)$.

We refer this to Hoyrup, Rojas (see also Franklin, Towsner "Randomness and non-ergodic systems" http://www.math.uconn.edu/ franklin/papers/ft-ergodic.pdf)

Sufficient condition for local stability

Corollary

If $f_n(\omega)$ a.s. effectively converges to $f(\omega)$ then a computable unbounded function $\rho(n)$ exists such that for any infinite sequence ω

$$K(\omega^n) \geq n - \rho(n) - O(1) \Longrightarrow \lim_{n\to\infty} f_n(\omega) = f(\omega).$$

logo

Scheme of proving local stability

$f_n(\omega) \to f(\omega)$ a.s. effectively converges $\Longrightarrow$

$f_n(\omega) \to f(\omega)$ for any Schnorr random $\omega \Longrightarrow$

$f_n(\omega) \to f(\omega)$ pointwise locally stable converges

Stable laws

- SLLN: $A_n(\omega) = \frac{1}{n}\sum_{i=1}^{n} \omega_i$ a.s. effectively converges to $\frac{1}{2}$.
- For any computable ergodic transformation preserving measure L and any computable bounded observable f, $E_n(\omega) = \frac{1}{n}\sum_{k=0}^{n-1} f(T^k\omega)$ a.s. effectively converges to $\int f dL$.

1) Item 1 follows from Chernoff inequality.
2) Item 2 follows from a generalization of maximal ergodic theorem by Galatolo, Hoyrup, Rojas "Computing the speed of convergence of ergodic averages and pseudorandom points in computable dynamical systems", EPTCS 24, 2010, pp. 718,

logo

Local stability property for ergodic case

Theorem

For any computable ergodic transformation T preserving the uniform measure and a computable bounded function f, a computable unbounded function $\alpha(n)$ exists such that

$$K(\omega^n) \geq n - \alpha(n) - O(1) \text{ for all } n \implies \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(T^i\omega) = \int f dL.$$

$\alpha(n)$ depends on T and f.

Uniform instability of the ergodic theorem

We cannot define such an $\alpha(n)$ common to all ergodic T and f.

Theorem

For any nondecreasing unbounded computable function $\alpha(n)$, a computable ergodic transformation T, a computable indicator function f, and a sequence $\omega \in \Omega$ exist such that

$$K(\omega^n) \geq n - \alpha(n) \text{ for all } n \text{ and}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i\omega) \text{ does not exist.}$$

logo

Instability property for non-ergodic transformations

Local stability property fails for non-ergodic transformations.

Theorem

A computable transformation T preserving the uniform measure exists such that for each unbounded computable function $\alpha(n)$ an infinite sequence $\omega \in \Omega$ exists such that

$$K(\omega^n) \geq n - \alpha(n) \text{ for all } n \text{ and}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i\omega) \text{ does not exist,}$$

for some computable indicator function f.

Transformation T is non-ergodic.

logo

Shannon – McMillan – Breiman theorem

$T(\omega_1\omega_2\dots) = \omega_2\omega_3\dots$ – left shift;

P – ergodic if it is invariant with respect to shift T.

An algorithmic version of the Shannon – McMillan – Breiman theorem holds Hochman (2009)):

Theorem

For any computable stationary ergodic measure P

$$K(\omega^n) \geq -\log P(\omega^n) - O(1) \Longrightarrow$$
$$\lim_{n\to\infty} \frac{K(\omega^n)}{n} = \lim_{n\to\infty} \frac{-\log P(\omega^n)}{n} = H,$$

where H is the entropy of the measure P.

Uniform instability property of the SMB theorem

Theorem

For any nondecreasing unbounded computable function $\alpha(n)$ and for any and $0 < \varepsilon < 1/4$ a computable stationary ergodic measure P with entropy $0 < H \leq \varepsilon$ and an infinite binary sequence ω exist such that

$$K(\omega^n) \geq -\log P(\omega^n) - \alpha(n) \textit{ for all } n,$$

$$\limsup_{n\to\infty} \frac{K(\omega^n)}{n} \geq \frac{1}{4}$$

and

$$\liminf_{n\to\infty} \frac{K(\omega^n)}{n} \leq \varepsilon.$$

Does a local stability holds for SMB theorem is an open problem.

Universal compression scheme

A code is a sequence of functions $\phi_n : \{0,1\}^n \to \{0,1\}^*$.
A code $\{\phi_n\}$ is called *universal* with respect to a class of stationary ergodic sources if for any computable stationary ergodic measure P (with entropy H_P)

$$\lim_{n\to\infty} \rho_{\phi_n}(\omega^n) = \frac{l(\phi_n(\omega^n))}{n} = H_P$$

almost surely, where $l(x)$ is length of a word x.

Lempel – Ziv coding scheme is an example of such universal coding scheme.

logo

Uniform instability property of any universal coding scheme

Theorem

For any unbounded nondecreasing computable function $\alpha(n)$ and $0 < \varepsilon < 1/4$ a computable stationary ergodic measure P with entropy $0 < H \leq \varepsilon$ exists such that for each universal code $\{\phi_n\}$ an infinite binary sequence ω exists such that

$$K(\omega^n) \geq -\log P(\omega^n) - \alpha(n) \text{ for all } n,$$

$$\limsup_{n\to\infty} \rho_{\phi_n}(\omega^n) \geq \frac{1}{4}$$

$$\liminf_{n\to\infty} \rho_{\phi_n}(\omega^n) \leq \varepsilon.$$

logo