Nonstandard Models of Arithmetic and Ramsey Theorem

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Main Theme

- This talk is on *C*ombinatorics, *C*omputability and *R*everse Mathematics.
- Motivations: Comparing relative strength of combinatorial principles; and study their logical consequences.
- The combinatorial principles in this talk will be related to Ramsey's Theorem.
- The strength and logical consequences are related to Computability and Reverse Math.

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Ramsey's Theorem

For $A \subseteq \mathbb{N}$, let $[A]^n$ denote the set of all n-element subsets of A.

Theorem (Ramsey (1930))

Any $f : [\mathbb{N}]^n \to \{0, 1, \ldots, k-1\}$ has an infinite homogeneous set $H \subseteq \mathbb{N}$*, namely, f is constant on $[H]^n$.*

We will loosely refer such an infinite homogeneous set as a "solution".

Notation: The version above is denoted by RT^n_k.

Our main focus is on RT^2_2 – Ramsey's Theorem for Pairs.

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One Proof of RT^2_2

Let f be a coloring of pairs, say *red* and *blue*.

- First step: Find an infinite subset $C \subseteq \omega$ on which f is "stable", i.e., for all x, $\lim_{y \in C, y \to \infty} f(x, y)$ exists.
- We call such a set C *cohesive* for f.
- Second step: One of
  $D^R = \{x \in C : x \text{ is "eventually red"}\}$ and
  $D^B = \{x \in C : x \text{ is "eventually blue"}\}$
  must be infinite, say D^R.
- Obtain a solution from D^R.

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COH and SRT^2_2

We extract two combinatorial principles out of the proof:

- Let R be an infinite set and $R^s = \{t|(s,t) \in R\}$. A set G is said to be *R-cohesive* if for all s, either $G \cap R^s$ is finite or $G \cap \overline{R^s}$ is finite.

- The cohesive principle COH states that for every R, there is an infinite G that is R-cohesive.

- SRT^2_2 states that every *stable* coloring of pairs has a solution.

- (Cholak, Jockusch and Slaman, 2001)

$$\mathsf{RT}^2_2 = \mathsf{COH} + \mathsf{SRT}^2_2.$$

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Motivating Questions

- How complicated is the homogeneous set H?
- Is COH or SRT^2_2 as strong as RT^2_2?
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Arithmetical Hierarchy

- Language of first order Peano Arithmetic: 0, S, $+$, $\times$, $<$; variables and quantifiers are intended for individuals.
- Formulas are classified by the number of alternating blocks of quantifiers: Σ^0_n and Π^0_n. (We always allow parameters.)
- We often talk about Δ^0_n formulas which have two equivalent forms, one Σ^0_n, one Π^0_n.
- Definable sets are classified by their defining formulas.
- (Slogan: "Computability is Definability": Recursive=Δ^0_1, and recursively enumerable sets = Σ^0_1 sets etc.)

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Fragments of First Order Peano Arithmetic

- Let $I\Sigma_n$ denote the induction schema for Σ^0_n-formulas; and $B\Sigma_n$ denote the Bounding Principle for Σ^0_n formulas.
- (Kirby and Paris, 1977) $\cdots \Rightarrow I\Sigma_{n+1} \Rightarrow B\Sigma_{n+1} \Rightarrow I\Sigma_n \Rightarrow \ldots$
- (Slaman, 2004) $I\Delta_n \Leftrightarrow B\Sigma_n$.
- (Note: When $n = 1$ we require the language has exponential function.)

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Fragments of Second Order Arithmetic

- Two-sorted language: (first order part) + variables and quantifiers for sets.
- RCA_0: Σ^0_1-induction and Δ^0_1-comprehension: For $\varphi \in \Delta^0_1$, $\exists X \forall n (n \in X \leftrightarrow \varphi(n))$.
- WKL_0: RCA_0 and every infinite binary tree has an infinite path.
- ACA_0: RCA_0 and for φ arithmetical, $\exists X \forall n (n \in X \leftrightarrow \varphi(n))$.
- (ATR_0 and Π^1_1-CA_0.) Π^1_1-formulas are of the form $\forall X \varphi$ where φ is arithmetical.

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Remarks on Axioms in Reverse Math

- They all assert the existence of certain sets.
- Some are measured by syntactical complexity, e.g. RCA_0 or ACA_0.
- Some are from the analysis of mathematical tools, e.g. WKL_0 corresponds to Compactness Theorem.

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Basic Models

- A *model* $\mathcal{M}$ *of second-order arithmetic* consists $(M, 0, S, +, \times, <, \mathcal{X})$ where $(M, 0, S, +, \times, <)$ is its first-order part and the set variables are interpreted as members of $\mathcal{X}$.

- Models of RCA_0: Its second-order part is closed under $\leq_T$ and Turing join, namely a *Turing ideal*.

- In the (minimal) model of RCA_0, $\mathcal{X}$ only consists of $\mathcal{M}$-recursive sets.

- (RCA_0 is the place to do constructive/finitary mathematics.)

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Remarks on Goals of Reversion

- Goal of Reverse Mathematics: What set existence axioms are needed to prove the theorems of ordinary, classical (countable) mathematics?
- Goal of Reverse Recursion Theory: What amount of induction are needed to prove the theorems of Recursion Theory, in particular, theorems about r.e. degrees.
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Rephrasing the Motivating Questions

- Question: Suppose f is recursive. What is the minimal syntactical complexity of a solution?
- Question: Which system in Reverse Mathematics does Ramsey's Theorem correspond? E.g., does RT_2^2 imply ACA_0?
- What are the first-order consequences of Ramsey's Theorem? E.g., does RT_2^2 imply $I\Sigma_2$?
- Does SRT_2^2 imply RT_2^2? In other words, if $\mathcal{X}$ contains solutions for all stable colorings, how about for general colorings?

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Earlier Results: (I)

Theorem (Jockusch, 1972)

1. *Every recursive coloring f has a Π^0_2 solution.*
2. *There is a recursive $f : [\mathbb{N}]^3 \to \{0, 1\}$ all of whose solutions compute $0'$.*
3. *There is a recursive coloring of pairs which has no Σ^0_2 solutions.*

Corollary

Over RCA_0*,*

$$\mathrm{ACA}_0 \Leftrightarrow \mathrm{RT}^3_2 \Leftrightarrow \mathrm{RT}^n_k.$$

$$\mathrm{ACA}_0 \Rightarrow \mathrm{RT}^2_2 \quad \textit{and} \quad \mathrm{WKL}_0 \not\Rightarrow \mathrm{RT}^2_2.$$

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Theorem (Hirst (1987))

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$$(\mathsf{S})\mathsf{RT}^2_2 \Rightarrow B\Sigma_2.$$

(This tells us a lower bound of its first order strength.)

Theorem (Seetapun and Slaman 1995)

There is a Turing ideal J such that $0' \notin J$ *and for every* $f : [\mathbb{N}]^2 \to \{0, 1\}$ *in J, there is a solution in J.*

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Earlier Results: (II)

Theorem (Hirst (1987))

Over RCA_0*,*

$$(\mathrm{S})\mathrm{RT}^2_2 \Rightarrow B\Sigma_2.$$

(This tells us a lower bound of its first order strength.)

Theorem (Seetapun and Slaman 1995)

There is a Turing ideal J *such that* $0' \notin J$ *and for every* $f : [\mathbb{N}]^2 \to \{0, 1\}$ *in* J*, there is a solution in* J*.*

Corollary

Over RCA_0*,*

$$(\mathrm{ACA}_0 \Rightarrow \mathrm{RT}^2_2 \quad \textit{and}) \quad \mathrm{RT}^2_2 \not\Rightarrow \mathrm{ACA}_0.$$

Conservation Results

- Harrington observed that WKL_0 is Π^1_1-conservative over RCA_0. i.e., any Π^1_1-statement that is provable in WKL_0 is already provable in RCA_0.
- Conservation results are used to measure the weakness of the strength of a theorem.

Theorem (Cholak, Jochusch and Slaman (2001))

RT^2_2 is Π^1_1-conservative over $\mathrm{RCA}_0 + I\Sigma_2$.

Corollary

Over RCA_0,

$$(\mathrm{RT}^2_2 \Rightarrow B\Sigma_2 \quad \textit{and}) \quad \mathrm{RT}^2_2 \not\Rightarrow I\Sigma_3.$$

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Combinatorics below RT^2_2

Hirschfeldt and Shore [2007], *Combinatorial principles weaker than Ramsey's theorem for pairs*.

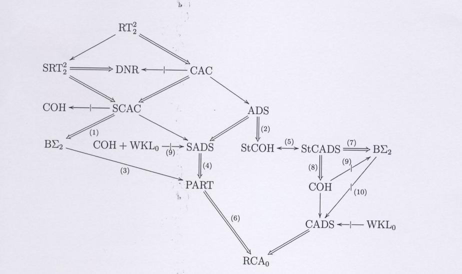

Diagram 4

In particular, COH does not imply RT^2_2.

Resent Results

Theorem (Jiayi Liu (2011))

Over RCA_0,

$$\mathsf{RT}^2_2 \not\rightarrow \mathsf{WKL}_0.$$

Theorem (Chong, Slaman and Y (2012))

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Remaining Questions and Obstacles

- Question 1: Over RCA_0, does SRT^2_2 imply RT^2_2?
- Question 2: Does SRT^2_2 imply $I\Sigma_2$? How about RT^2_2?
- Attempt for Q 1: Show that stable colorings always have a low solution. Or equivalently, every Δ^0_2-set contains or is disjoint from an infinite low set.

Theorem (Downey, Hirschfeldt, Lempp and Solomon (2001))

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Chong (2005): We should look at nonstandard models of fragments of arithmetic, because:

- DFLS theorem is done on ω, whose proof involves infinite injury method thus requires $I\Sigma_2$.
- There is a model of $B\Sigma_2$ but not $I\Sigma_2$ in which every incomplete Δ^0_2 set is low.

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Technical Remarks: A Tailor-Made Model

- It is countable and its first order part satisfies $PA^{-} + B\Sigma_2$ but not $I\Sigma_2$.
- ω is a Σ^0_2-cut and there is a Σ^0_2 function $g : \omega \to M$ which is unbounded.
- $\mathcal{M} = \bigcup_{n \in \omega} \mathcal{M}_n$ is a union of chains such that $\mathcal{M}_n$ satisfies full Peano arithmetic.
- Σ^0_1-reflection property: For each $n \in \omega$, $\mathcal{M}_n \prec_1 \mathcal{M}$;
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Given a Δ_2^0 set A, we construct an infinite G subset of either A or $\overline{A}$, such that $\emptyset'$ can determine the Σ_1-theory of G.

- Blocking method: We divide the whole Σ_1-theory of G into ω many blocks: $B_n = \{\varphi_e(G) : e \leq g(n)\}$ where $\{\varphi_e : e \in M\}$ is a fixed enumeration of $\Sigma_1^0(G)$ sentences.
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More Resent Results

Theorem (Chong, Slaman and Y (ta2))

$$\mathsf{RT}^2_2 \not\Rightarrow I\Sigma_2.$$

- We knew how to satisfy COH and SRT^2_2 individually without satisfying $I\Sigma_2$.
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- Question: What happens in ω-model? (Kind of "provability vs. truth" question.)

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